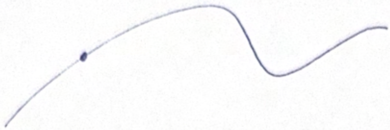
MATH 2010

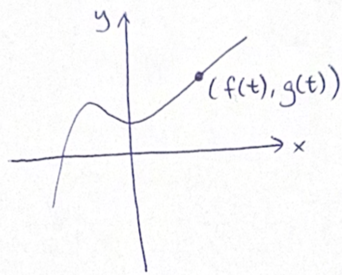
Tutorial 3

Curves



$$\vec{x}(t) = (x_1(t), \dots, x_n(t)) \in \mathbb{R}^n \quad , \; t \in \mathbb{R}$$

On the xy-plane :



$x = f(t)$,
$y = g(t)$.

Exercises 10.2

Q15 Assuming that the equations define x and y implicitly as differentiable functions $x = f(t)$, $y = g(t)$, find the slope of the curve $x = f(t)$, $y = g(t)$ at the given value of t.

$$x^3 + 2t^2 = 9 \; , \; 2y^3 - 3t^2 = 4 \; , \; t = 2$$

$$\boxed{\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \qquad \left(\because \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}\right)$$

- $x^3 + 2t^2 = 9$ (Implicit Differentiation , w.r.t. t)

$$3x^2 \frac{dx}{dt} + 4t = 0$$

$$\frac{dx}{dt} = \frac{-4t}{3x^2}$$

- $2y^3 - 3t^2 = 4$

$$6y^2 \frac{dy}{dt} - 6t = 0$$

$$\frac{dy}{dt} = \frac{6t}{6y^2} = \frac{t}{y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{t}{y^2}}{\frac{-4t}{3x^2}} = \frac{3x^2}{-4y^2}$$

At $t=2$: $x^3 + 2(2)^2 = 9 \Rightarrow x = 1$

$2y^3 - 3(2)^2 = 4 \Rightarrow y = 2$

$$\text{Slope} = \left.\frac{dy}{dx}\right|_{(1,2)} = \frac{3}{-4(2)^2} = -\frac{3}{16}$$

Q25 Find the length of the curve

$x = \cos t$, $y = t + \sin t$, $0 \le t \le \pi$.

Let $\vec{x}(t) = (x_1(t), \dots, x_n(t))$.

$\vec{x}'(t) = (x_1'(t), \dots, x_n'(t))$ tangent vector.

Arclength of $\vec{x}(t)$ for $a \le t \le b$ is

$$S = \int_a^b \|\vec{x}'(t)\| \, dt$$

$$\vec{x}'(t) = (-\sin t, 1 + \cos t)$$

$$\|\vec{x}'(t)\| = \sqrt{\sin^2 t + (1+\cos t)^2}$$

$$= \sqrt{\sin^2 t + 1 + 2\cos t + \cos^2 t}$$

$$= \sqrt{2 + 2\cos t}$$

$$\text{Length} = \int_0^\pi \sqrt{2 + 2\cos t} \, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{1 + \cos t} \, dt$$

$$= \sqrt{2} \int_0^\pi \sqrt{(1+\cos t)\frac{1-\cos t}{1-\cos t}} \, dt$$

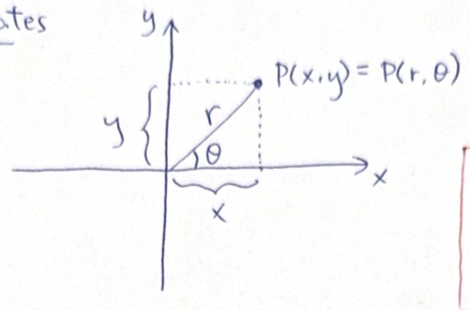
$$= \sqrt{2} \int_0^\pi \sqrt{\frac{1-\cos^2 t}{1-\cos t}} \, dt = \sqrt{2} \int_0^\pi \sqrt{\frac{\sin^2 t}{1-\cos t}} \, dt$$

$$= \sqrt{2} \int_0^\pi \frac{\sin t}{\sqrt{1-\cos t}} \, dt \quad (\because \sin t \ge 0 \text{ on } [0,\pi])$$

$$= \sqrt{2} \int_0^\pi \frac{1}{\sqrt{1-\cos t}} \, d(1-\cos t) = \sqrt{2}\left(\left.\frac{(1-\cos t)^{\frac{1}{2}}}{\frac{1}{2}}\right|_0^\pi\right) = \sqrt{2}(2)(\sqrt{2}-0)$$

$$= 4$$

Polar coordinates



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ r^2 &= x^2+y^2 \end{aligned}}$$

Exercises 10.3

Q43 Replace the polar equation with equivalent Cartesian equation. Then describe or identify the graph.

$$r^2 + 2r^2\cos\theta\sin\theta = 1$$

$$r^2 + 2(r\cos\theta)(r\sin\theta) = 1$$

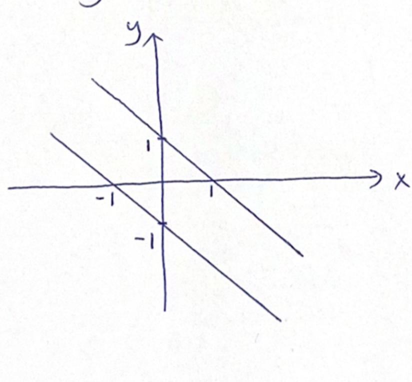
$$x^2+y^2+2(x)(y) = 1$$

$$(x+y)^2 = 1$$

$$x+y = \pm 1$$

$$y = \pm 1 - x$$

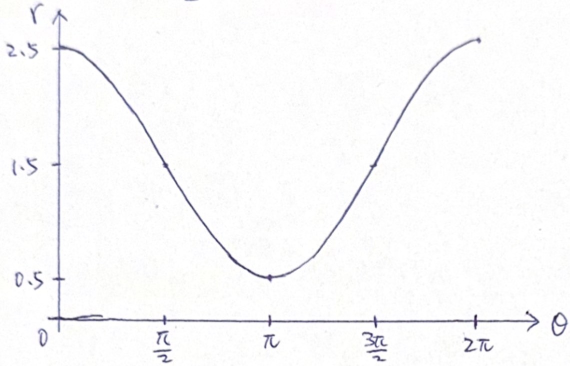
two straight lines of slope -1 and y-intercepts ± 1

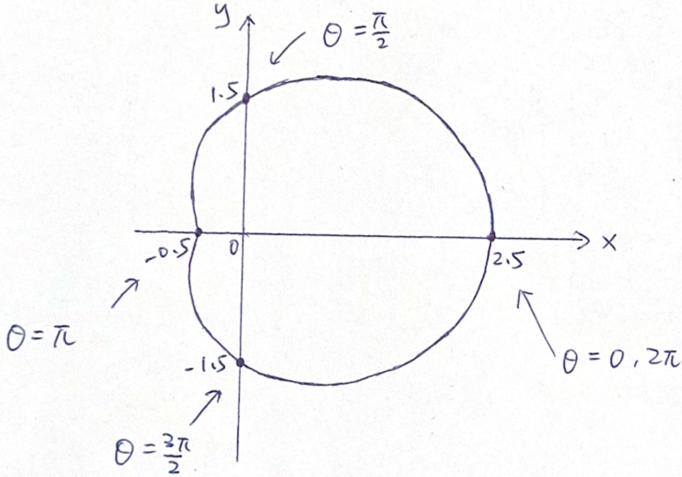


Graphing Polar Coordinate Equations

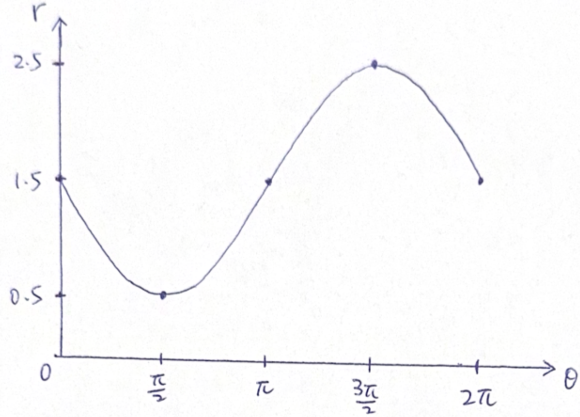
Exercises 10.4

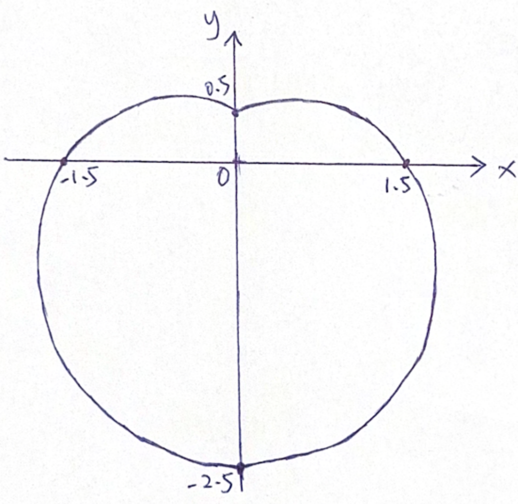
Q27 a. $r = \frac{3}{2} + \cos\theta$





b. $r = \frac{3}{2} - \sin\theta$





Exercises 12.3

Q1 Find the curve's unit tangent vector. Also, find the length of the indicated portion of the curve.

$$\vec{r}(t) = (2\cos t)\vec{i} + (2\sin t)\vec{j} + \sqrt{5}\,t\,\vec{k}, \quad 0 \le t \le \pi$$

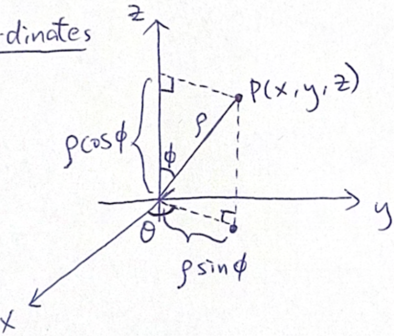
$$\vec{r}'(t) = (-2\sin t,\ 2\cos t,\ \sqrt{5})$$

$$\|\vec{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 5} = \sqrt{4+5} = 3$$

$$\text{unit tangent vector} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \left(-\tfrac{2}{3}\sin t,\ \tfrac{2}{3}\cos t,\ \tfrac{\sqrt{5}}{3}\right)$$

$$\text{Length} = \int_0^{\pi} \|\vec{r}'(t)\|\,dt = \int_0^{\pi} 3\,dt = 3\pi$$

Spherical Coordinates



$$\boxed{\begin{aligned} x &= \rho\sin\phi\cos\theta \\ y &= \rho\sin\phi\sin\theta \\ z &= \rho\cos\phi \\ \rho^2 &= x^2+y^2+z^2 \end{aligned}}$$

Example) Find a spherical coordinate equation for the sphere

$$x^2 + y^2 + (z-1)^2 = 1$$

$$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\rho^2 - 2\rho\cos\phi = 0$$

$$\rho^2 = 2\rho\cos\phi$$

$$\rho = 2\cos\phi$$