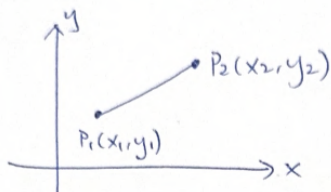


Distance

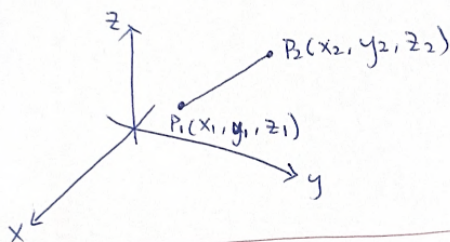
In 2-dimensional xy -plane :



Distance between $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In 3-dimensional space ($\mathbb{R}^3$) :



Distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

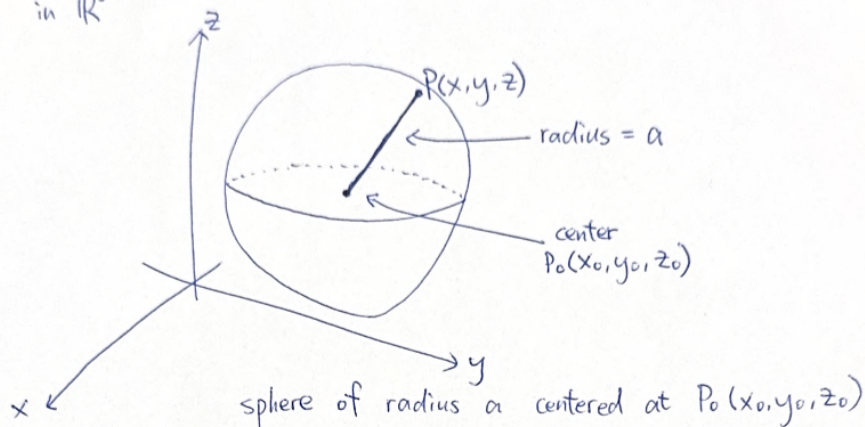
Example : The distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$ is

$$|P_1 P_2| = \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2}$$

$$= \sqrt{16 + 4 + 25}$$

$$= \sqrt{45} \approx 6.708$$

Sphere in $\mathbb{R}^3$



For any point $P(x, y, z)$ on the sphere,

Distance between $P(x, y, z)$ and center $P_0(x_0, y_0, z_0) = a$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = a$$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = a^2$$

The standard Equation for

the sphere of radius a and center (x_0, y_0, z_0)

Example: Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0.$$

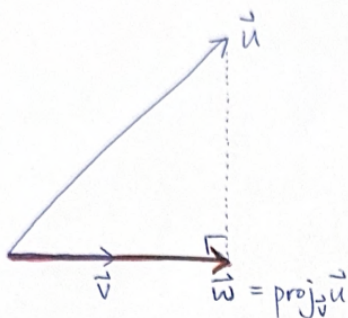
$$(x^2 + 3x) + y^2 + (z^2 - 4z) = -1$$

$$(x^2 + 3x + (\frac{3}{2})^2) + y^2 + (z^2 - 4z + (\frac{4}{2})^2) = -1 + (\frac{3}{2})^2 + (\frac{4}{2})^2$$

$$(x + \frac{3}{2})^2 + y^2 + (z - 2)^2 = \frac{21}{4}$$

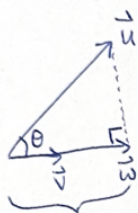
The center is $(-\frac{3}{2}, 0, 2)$ and the radius is $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

Projection



vector projection of $\vec{u}$
onto a nonzero vector $\vec{v}$

Notation: $\text{proj}_{\vec{v}} \vec{u}$



θ : angle between
 $\vec{u}$ and $\vec{v}$

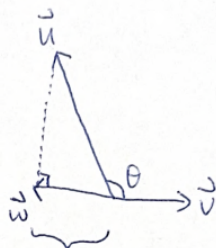
$$\text{length} = \|\vec{u}\| \cos \theta$$

$$\text{direction} = \frac{\vec{v}}{\|\vec{v}\|}$$

$$\vec{w} = \underbrace{(\|\vec{u}\| \cos \theta)}_{\text{length}} \underbrace{\left(\frac{\vec{v}}{\|\vec{v}\|}\right)}_{\substack{\text{direction} \\ \text{(unit vector)}}}$$

$$= \|\vec{u}\| \underbrace{\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}_{\cos \theta} \frac{\vec{v}}{\|\vec{v}\|}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$



$$\text{length} = -\|\vec{u}\| \cos \theta$$

$$\text{direction} = -\frac{\vec{v}}{\|\vec{v}\|}$$

The number $\|\vec{u}\| \cos \theta$ (could be positive or negative)
is called the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

The vector projection of $\vec{u}$ onto $\vec{v}$ is

$$\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \right) \left(\frac{\vec{v}}{\|\vec{v}\|} \right)$$

The scalar component of $\vec{u}$ in the direction of $\vec{v}$ is

$$\|\vec{u}\| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} = \vec{u} \cdot \frac{\vec{v}}{\|\vec{v}\|}$$

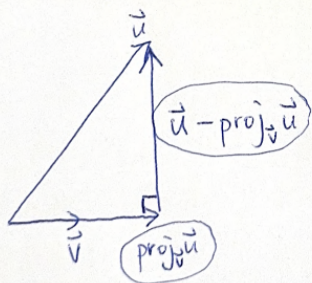
Example: Find the vector projection of $\vec{u} = 6\vec{i} + 3\vec{j} + 2\vec{k}$ onto $\vec{v} = \vec{i} - 2\vec{j} - 2\vec{k}$ and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

$$\begin{aligned} \text{proj}_{\vec{v}} \vec{u} &= \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} = \frac{6-6-4}{1+4+4} (\vec{i} - 2\vec{j} - 2\vec{k}) \\ &= -\frac{4}{9} (\vec{i} - 2\vec{j} - 2\vec{k}) = -\frac{4}{9} \vec{i} + \frac{8}{9} \vec{j} + \frac{8}{9} \vec{k} \end{aligned}$$

Scalar component of $\vec{u}$ in the direction of $\vec{v}$

$$\begin{aligned} &= \|\vec{u}\| \cos \theta = \vec{u} \cdot \frac{\vec{v}}{\|\vec{v}\|} = (6\vec{i} + 3\vec{j} + 2\vec{k}) \cdot \left(\frac{1}{3} \vec{i} - \frac{2}{3} \vec{j} - \frac{2}{3} \vec{k} \right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3} \end{aligned}$$

Example: Verify that the vector $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to the projection vector $\text{proj}_{\vec{v}} \vec{u}$.



$\therefore$ The vector $\text{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v}$ is parallel to $\vec{v}$.

$\therefore$ It suffices to show that the vector $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$.

$$\begin{aligned}
 & (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \cdot \vec{v} \\
 &= \left(\vec{u} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \right) \cdot \vec{v} \\
 &= \vec{u} \cdot \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v} \right) \cdot \vec{v} \\
 &= \vec{u} \cdot \vec{v} - \underbrace{\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} (\vec{v} \cdot \vec{v})}_{\text{scalar}} \\
 &= \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \|\vec{v}\|^2 \\
 &= 0
 \end{aligned}$$

The vector $\vec{u}$ is expressed as a sum of two orthogonal vectors.

$$\begin{aligned}
 \vec{u} &= \text{proj}_{\vec{v}} \vec{u} + (\vec{u} - \text{proj}_{\vec{v}} \vec{u}) \\
 &= \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} + \left(\vec{u} - \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \right) \vec{v} \right)
 \end{aligned}$$

