Triple product (only in $\mathbb{R}^3$)

let $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$, the triple product of $\vec{a}, \vec{b}$ & $\vec{c}$ (order) is important is defined by $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

Why we should mention this $(\vec{a} \times \vec{b}) \cdot \vec{c}$, because this formula has these important

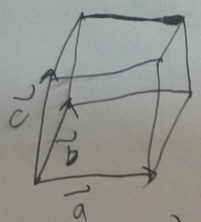
properties ① $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

I tell you the key oberservation.

abc $\Leftrightarrow$ bca $\Leftrightarrow$ cab

② Geometric meaning

$|(\vec{a} \times \vec{b}) \cdot \vec{c}| =$ Volume of the parallelopiped spanned by $\vec{a}, \vec{b}$ & $\vec{c}$

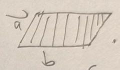


③ $(\vec{a} \times \vec{b}) \cdot \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

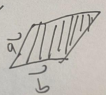
① is a simple consequen of ①, because

$$(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (\vec{b}\times\vec{c})\cdot\vec{a}$$

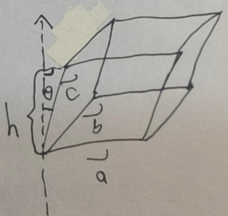
We only need to prove ② and ③, now we prove ②!

Recall the geometric meaning of $a\times b$, it is area of .

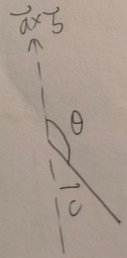
$$(\vec{a}\times\vec{b})\cdot\vec{c} = \underbrace{\|(\vec{a}\times\vec{b})\|}_{\text{Area of}} \cdot \vec{c}_{\text{projection onto } \vec{a\times b}} = \text{Volume of parallelopiped}$$



More explanation:



for $0 \le \theta < \frac{\pi}{2}$

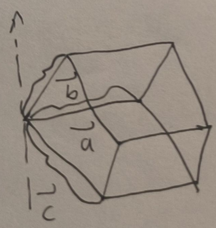


The idea is

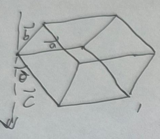
① $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |(\vec{b}\times\vec{a})\cdot\vec{c})$

② $\sin(\pi-\theta) = -\sin(-\theta) = \sin\theta$

For $\frac{\pi}{2} < \theta \le \pi$



So now we consider

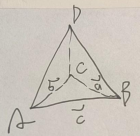


, where $\theta' = \pi - \theta$, because ① $|(\vec{a}\times\vec{b})\cdot\vec{c}| = |(\vec{b}\times\vec{a})\cdot c|$ ② $\sin(\pi-\theta) = \sin\theta$

Some other topics for $(\vec{a}\times\vec{b})\cdot\vec{c}$,

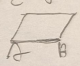
(i) $(\vec{a}\times\vec{b})\cdot\vec{c} = 0 \Leftrightarrow \mathrm{Vol}(\text{Parallelepiped}) = 0 \Leftrightarrow \{\vec{a},\vec{b},\vec{c}\}$ are linearly dependent.

(ii) Tetrahedron



$\mathrm{Vol}(\text{Tetrahedron})$

$= \frac{1}{3}\,\mathrm{Area}(\triangle ABC)\cdot \text{height}$

$= \frac{1}{3}\cdot\frac{1}{2}\,\mathrm{Area}($  $)$

$= \frac{1}{6}\,\mathrm{Vol}(\text{Parallelepiped})$

$= \frac{1}{6}\,|(\vec{a}\times\vec{b})\cdot\vec{c}|$.

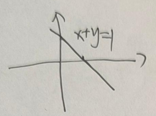
eg. Let $A = (1,0,1)$, $B = (1,1,2)$, $C = (2,1,1)$, $D = (2,1,3)$, find Volume of tetrahedron ABCD.

$\vec{a} = \overrightarrow{AB} = (1,1,2) - (1,0,1) = (0,1,1)$, $\vec{b} = \overrightarrow{AC} = (2,1,1) - (1,0,1) = (1,1,0)$, $\vec{c} = \overrightarrow{AD} = (2,1,3) - (1,0,1) = (1,1,2)$

$\mathrm{vol}(\text{Tetrahedron}) = \frac{1}{6}|(\vec{a}\times\vec{b})\cdot\vec{c}| = \frac{1}{6}\left|\det\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\right| = \frac{1}{6}|-2| = \frac{1}{3}$.

Linear objects in R^n (Basic for the high-dimension calculus!) Content in differential geometry
(lines, plane, k-plane, hyperplane)

Line in R^2 $x+y=1$



But we have three ways to represent the equation of the line in R^3!

(1) General form (intersection of two planes)

$$\begin{cases} A_1x+B_1y+C_1z+D_1=0, \longrightarrow \text{the equation of the plane} \\ A_2x+B_2y+C_2z+D_2=0. \end{cases}$$

Question: why $x+y=1$ is not this form?

Because this is general expression in R^3. We can also let it be

$$\begin{cases} x+y+z=1 \\ z=0 \end{cases}$$

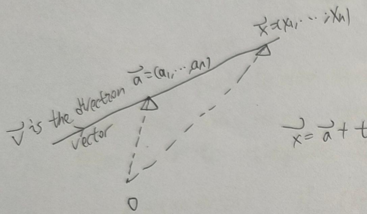
(2) Symmetric equations (point-direction form)

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$$

(3) Parametric equations (vector-form)

$$r(t)=r_0+tV \quad \text{or} \quad \begin{cases} x=x_0+at, \\ y=y_0+at, \\ z=z_0+at, \end{cases}$$ V is a directional vector

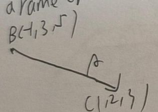
Some more explanation of the parametric form: (a picture to let you understand the definition better!)



$\vec{x}=\vec{a}+t\vec{v}$, $t\in\mathbb{R}$ is called a parameter.

Example: ① A line passes through $A=(1,2,3)$ and $B(-1,3,5)$

(1) Parametric equations



$$\vec{OA}+t\vec{AB}$$
$$=(1,2,3)+t(-2,1,2).$$

(2) Symmetric equations

$$\frac{x-1}{-2}=\frac{y-2}{1}=\frac{z-3}{2}=t$$

} help you to understand the symmetric equations.

② From Two-planes form to the parametric form

$$\begin{cases} x+y+6z=6 \\ x-y-2z=-2 \end{cases} \Rightarrow (0,0,1) \text{ and } (1,2,\tfrac{1}{2})$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (0,0,1)+t\begin{pmatrix} 1 \\ 2 \\ -\frac{1}{2} \end{pmatrix}.$$

Parametric form, a point and two vector

After introducing the equation of the line, we should talk about the equation of the plane.

We have five ways to express one plane,

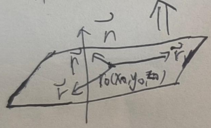
1. Point-Normal form

Given one point on the plane and the normal vector of the plane.
If we have the normal vector, we can get infinite plane which is perpendiculer to theplane, If we have one point, we can get the result.

$r_0 = (x_0, y_0, z_0)$, $\vec{n} = (A, B, C)$, the plane is

$\vec{n} \cdot (r - r_0) = 0$ or $(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$

$\Leftrightarrow A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$



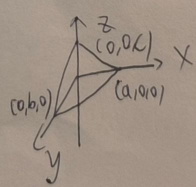
2. General form

See the last part in the point-normal form,

$Ax + By + Cz + D = 0$

3. Intercept form,

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$



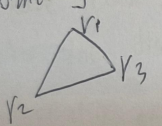
parametric form, a point and two vectors not in the same or opposite direction.

If we have two vectors, we can get a plane with infinite planes which are parallel to the each other. If have one point, we can ensure the position of the plane.

$$r(s,t) = r_0 + s\vec{u} + t\vec{v},$$ where s and t are parameters.

(5) Three-point form

$$(r-r_1)\cdot\underbrace{[(r_2-r_1)\times(r_3-r_1)]}_{\text{give the normal vector}} = 0$$



just like the point-normal form.

Example: we have three points $A=(0,0,1)$, $B=(0,2,0)$ and $C=(-1,1,0)$.

$$\overrightarrow{BA} = (0,2,0)-(0,0,1) = (0,2,-1)$$

and

$$\overrightarrow{CA} = (-1,1,0)-(0,0,1) = (-1,1,-1)$$

The parametric form is

$$r(s,t) = (0,0,1) + \alpha(0,2,-1) + \beta(-1,1,-1).$$

(ii) point-normal form

$$\vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & -1 \\ -1 & 1 & -1 \end{vmatrix} = (-1)^{1+1}\hat{i}\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - (-1)^{1+2}\hat{j}\begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} + (-1)^{1+3}\hat{k}\begin{vmatrix} 0 & 2 \\ -1 & 1 \end{vmatrix}$$

$$= -\hat{i} + \hat{j} + 2\hat{k} = (-1, 1, 2)$$

So we have $((x, y, z) - (0, 0, 1)) \cdot (-1, 1, 2) = 0$

Example. find the distance between a point and a plane.

$A = (2, 1, 1)$ and the plane $P: -x + 2y - z = -4$ — (*)

By the general form, we have $(-1, 2, -1)$ is a normal vector.

So a point (x, y, z) in the plane

$$\begin{cases} -x + 2y - z = -4 \\ \dfrac{(2-x)}{-1} = \dfrac{1-y}{2} = \dfrac{1-z}{-1} \end{cases}$$

$\Rightarrow$ $4 - 2x = y - 1$ and $2 - x = 1 - z$

$\Rightarrow$ $5 - 2x = y$ and $z = x - 1$

$\Rightarrow$ $-x + 10 - 4x - x + 1 = -4 \Rightarrow -6x = -15 \Rightarrow x = \frac{5}{2} \Rightarrow y = 0,\ z = \frac{3}{2}$

We only need the distance between $(\frac{5}{2}, 0, \frac{3}{2})$ and $(2,1,1)$,

$$\text{distance} = \sqrt{(2-\frac{5}{2})^2 + (1-0)^2 + (1-\frac{3}{2})^2} = \frac{\sqrt{6}}{2}.$$

General formula: a point (x_0, y_0, z_0) to the plane $Ax+By+Cz+D=0$ is

$$d = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}$$

—我—是—分—割—线—

After introducing the line and plane in the high-dimensional space, now we will introduce several other concepts, for example, curves!

Curves in $\mathbb{R}^n$:

Defn: Let $I \subset \mathbb{R}$ be an interval, a (continuous) curve in $\mathbb{R}^n$ is a continuous (vector-valued) function

$$\vec{x}: I \to \mathbb{R}^n$$

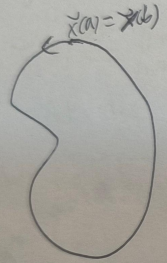
i.e. $t \in I$, $\vec{x}(t) = (x_1(t), \cdots, x_n(t)) \in \mathbb{R}^n$ such that every component function $x_i(t)$ is continuous $(i=1,\cdots,n)$.

Defn: A curve $\boxed{\vec{x}:[a,b]\to\mathbb{R}^n}$ is said to be

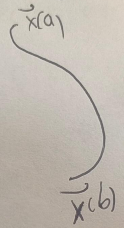
(i) <u>Closed</u> if $\vec{x}(a)=\vec{x}(b)$

(ii) <u>simply</u> if $\vec{x}(t_1)\neq\vec{x}(t_2)$ for $a\le t_1<t_2\le b$ (except possibly at $\underbrace{\vec{x}(a)=\vec{x}(b)}_{\text{closed}}$.

Closed & simply

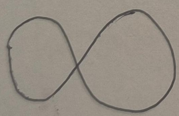


not Closed & simple



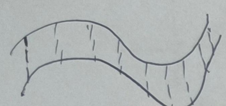
Closed & not simply

↓ infinity!



Euclidean space x, y, z $(x^2+y^2+z^2=1)$

Sometime we need consider the integral over a curve, so we need a method to describe the change in a short time.



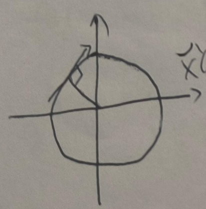
Definition, Tangent vector, $\vec{x}'(a)$ = tangent vector of $\vec{x}(t)$ at $t=a$.

(1) $\vec{x}'(t) \overset{def}{=\!=} \lim_{h\to 0} \frac{\vec{x}(t+h)-\vec{x}(t)}{h} = (x_1'(t), \cdots, x_n'(t))$

if the limited exists,

(2) $\lim_{t\to a} \vec{x}(t) = (\lim_{t\to a} x_1(t), \cdots, \lim_{t\to a} x_n(t))$.

Example: $\vec{x}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$).



$\vec{x}'(t) = \lim_{h\to 0} \frac{(\cos(t+h)-\cos t, \sin(t+h)-\sin t)}{h} = (-\sin t, \cos t)$

We also have $\vec{x}(t) \perp \vec{x}'(t)$.

Arclength of a curves

Rules for the derivative of the parameter of the curve

(1) $(\vec{x}(t)+\vec{y}(t))' = \vec{x}'(t)+\vec{y}'(t)$

(2) $(c\vec{x}(t))' = c\,\vec{x}'(t)$

(3) $(f(t)\vec{x}(t))' = f'(t)\vec{x}(t)+f(t)\vec{x}'(t)$

(4) $(\vec{x}(t)\cdot\vec{y}(t))' = \vec{x}'(t)\cdot\vec{y}(t)+\vec{x}(t)\cdot\vec{y}'(t)$

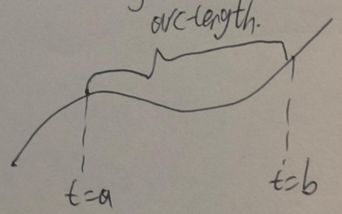
(5) $(\vec{x}(t)\times\vec{y}(t))' = \vec{x}'(t)\times\vec{y}(t)+\vec{x}(t)\times\vec{y}'(t)$

After the tangent vector, when we have the tangent vector, we can have the arc-length.

$$S=\int_a^b \|\vec{x}'(t)\|dt$$

Q1

HW 1: Prove arc-length formula



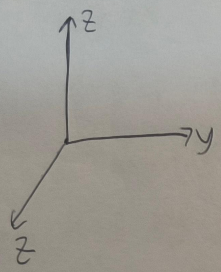
Arc length {of a curves

Let $\vec{x}(t)$ be a curve with $\vec{x}'(t)$ exists and continuous.

$$\boxed{\text{Define: Arc length of } \vec{x}(t) \text{ for } a \le t \le b \text{ is } \quad S = \int_a^b \|\vec{x}'(t)\|\, dt}$$

How to prove? too complicated

eg: (Helix) $\vec{x}(t) = (\cos t, \sin t, t)$ (curve in $\mathbb{R}^3$) $t \in [0, 2\pi]$



$\vec{x}'(t) = (-\sin t, \cos t, 1)$

arclength: $\|\vec{x}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{2}$,

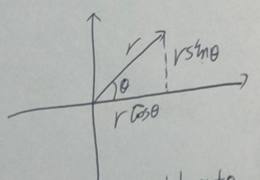
$$S = \int_0^{2\pi} \|\vec{x}'(t)\|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi.$$

Eu... space x, y, z $(x^2+y^2+z^2...)$ In $f(x+h)-h$

After introducing the parameter, we will introduce the polar coordinates in R^2.

$p=(x,y)=(r\cos\theta, r\sin\theta)$,

$r\in[0,\infty)$, $\theta\in[0,2\pi)$



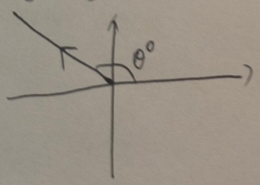
Change of coordinate from $(x,t) \Leftrightarrow (r,\theta)$

$$\begin{cases} x=r\cos\theta, \\ y=r\sin\theta \end{cases} \& \begin{cases} r=\sqrt{x^2+y^2}, \\ \theta=\tan^{-1}\left(\frac{y}{x}\right) \end{cases}$$

eg: $x^2+y^2=1 \Leftrightarrow r^2=1$ (which means $r=1$)

So $\vec{x}(t)=(r\cos t, r\sin t)$

eg: Half ray from the origion



$$\begin{cases} \theta=\theta_0 \\ r=t \end{cases}$$

So $\vec{x}(t)=\begin{cases} r=t \\ \theta=\theta_0. \end{cases} \Rightarrow (t\cos\theta_0, t\sin\theta_0)$

x, y, z

Fuct

Archimedes spiral

~~@@@~~ $r = k\theta \Rightarrow$ Parametric form

$$\begin{cases} x = r\cancel{\sin\theta}\cos\theta = k\theta\cancel{\sin\theta}\cos\theta \\ y = r\cancel{\cos\theta}\sin\theta = k\theta\cancel{\cos\theta}\sin\theta \end{cases}$$

eg: hyberpolic equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad x = r\cos\theta, \; y = r\sin\theta$$

$$\Rightarrow r^2\left(\frac{\cos\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) = 1$$

So
$$\begin{cases} x = r\cos\theta = \sqrt{\frac{\cos^2\theta}{a^2} - \frac{\sin\theta}{b^2}}\cos\theta \\ y = r\sin\theta = \sqrt{\frac{\cos\theta}{a^2} - \frac{\sin\theta}{b^2}}\sin\theta \end{cases}$$

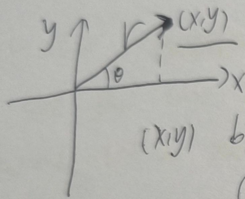
We can always expree the hyberbolic equation as parameter form, θ is nothing but a parameter. ~~the question is off~~

Euclidean space x, y, z $x^2+y^2+z^2=1$

$\lim_{h \to 0} \frac{f(x+h)-h}{h}$

$\sin x$

① polar coordinate in R^2



(x,y) by (r,θ)

$(r\cos\theta, r\sin\theta)$

change of coordinate form $(x,t) \Leftrightarrow (r,\theta)$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ & $\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{cases}$

$x = r\cos\theta = r(\theta)\cos\theta$

$y = r\sin\theta$

eg: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\boxed{\begin{matrix} x = r\cos\theta \\ y = r\sin\theta \end{matrix}}$

$\Rightarrow r^2\left(\frac{\cos\theta}{a^2} - \frac{\sin\theta}{b^2}\right) = 1$

...)) ∪ [$2\pi - \cos^{-1}(\frac{1}{a})$, 2π] due to r

Negative r

Our convention is $r \geq 0$. But sometimes is convenient to allow $r<0$ by the interpretation

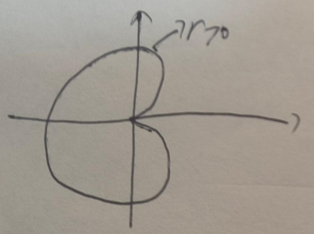
$(x,y) = (r\cos\theta, r\sin\theta) = (-|r|\cos\theta, -|r|\sin\theta) = (|r|\cos(\theta+\pi), |r|\sin(\theta+\pi))$.

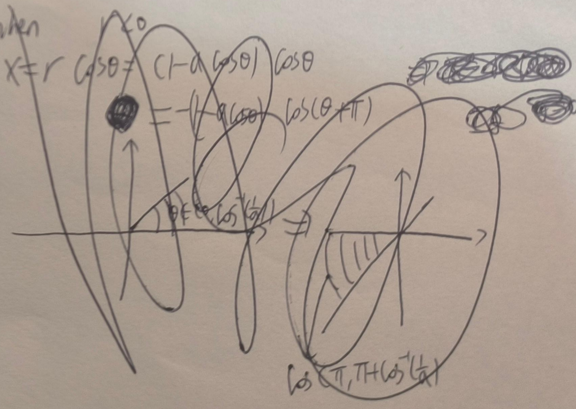
$r = 1 - a\cos\theta$. $a > 1$.

We will find this situation is $r<0$ sometime. You may think this is hard to understand. Let us see the picture it describes.

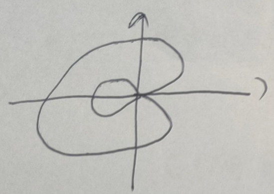
$r \geq 0 \Rightarrow 1 - a\cos\theta \geq 0 \Rightarrow \cos\theta \leq \frac{1}{a} < 1 \Rightarrow \theta \in [\cos^{-1}(\frac{1}{a}), 2\pi - \cos^{-1}(\frac{1}{a})]$

$r<0 \Rightarrow \theta \in [0, \cos^{-1}(\frac{1}{a})) \cup (2\pi - \cos^{-1}(\frac{1}{a}), 2\pi]$





When $r<0$, no image will appear on $[0, \cos^{-1}(\frac{1}{a})) \vee [2\pi - \cos^{-1}(\frac{1}{a}), 2\pi]$ due to r is negative, so it is

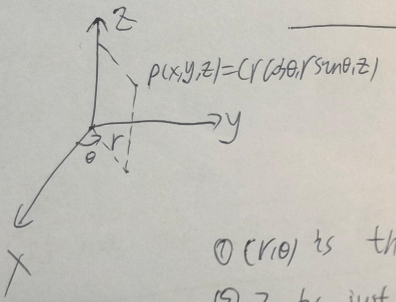


It is call Limaçon curve.

$$(x^2+y^2+ax) = (x^2+y^2)$$

This negative r means $\theta+\pi$, to represent a rotation!

Other special coordinate, <u>cylindrical coordinate</u>



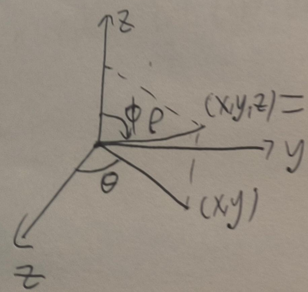
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\begin{cases} x^2+y^2 \le r^2 \\ 0 \le z \le y \end{cases}$$

① (r,θ) is the polar coordinate for xy-plane
② z is just z-axis

Special coordinates
$$\begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi. \end{cases}$$

$\Rightarrow x^2+y^2+z^2 = r^2$



$(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

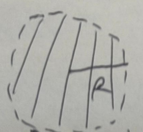
Some Topological concepts in R^n:

open ball: $B_R(x_0)=\{x\in R^n, |x-x_0|<R\}$ is the open ball of the radius R and centered at x_0.

Closed ball: $\overline{B_R(x_0)}=\{x\in R^n, |x-x_0|\le R\}$ is the closed ball of the radius R and centered at x

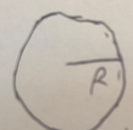
① $\overline{B_R(x_0)}$ is called the closure of $B_R(x_0)$.

If $n=2$, the ball is called the disk!



open ball

The dotted line means not included



Closed ball

② For a closed ball $\overline{B_R(x_0)}$, $B_R(x_0)$ is called the interior of the $\overline{B_R(x_0)}$.

③