Lecture 12:

Image denoising by solving Anisotropic heat diffussion

Consider the PDE:

$$(*) \quad \frac{\partial I(x,y,t)}{\partial t} = t\left[\frac{\partial^2 I(x,y,\ )}{\partial x^2} + \frac{\partial^2 I(x,y,t)}{\partial y^2}\right] = t\,\nabla\cdot(\nabla I)$$

($\nabla\cdot$ = divergence ; $\nabla\cdot(v_1, v_2) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$) ($\nabla I = \left(\frac{\partial I}{\partial x}, \frac{\partial I}{\partial y}\right)$)

Then: $g(x,y,t) = \frac{1}{2\pi t^2} e^{-(x^2+y^2)/2t^2}$ satisfies $(*)$.

Observation: We'll see that Gaussian filter is approximately solving $(*)$

Gaussian filter = convolution of I with the Gaussian function:

$$\tilde{I}(x,y,t) = I * g(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-u, y-v)\, I(u,v)\, du\, dv$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v;t)\, I(x-u, y-v)\, du\, dv$$

(Analogous to discrete convolution)

$$\therefore \frac{\partial \tilde{I}}{\partial t} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial g(u,v,t)}{\partial t} I(x-u, y-v)\, du\, dv$$

$$= t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 g(u,v;t)}{\partial u^2} I(x-u, y-v)\, du\, dv + t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2 g}{\partial v^2}(u,v;t)\, I(x-u,y-v)\, du\, dv$$

$$= t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v;t) \frac{\partial I}{\partial x^2}(x-u, y-v)\, du\, dv + t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v;t) \frac{\partial I}{\partial y^2}(x-u, y-v)\, du\, dv$$

$$= t\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v;t)\, I(x-u, y-v)\, du\, dv.$$

$$= t\, \nabla\cdot(\nabla \tilde{I})(x,y,t)$$

$\therefore$ Gaussian filter = solving "diffussion" eqt !!

Diffussion equation can be used for image denoising.

Consider a sequence of images:

$$I^0,\ I^1,\ I^2,\ \dots,\ I^k,\ I^{k+1},\ \dots \in M_{N\times N}(\mathbb{R})$$

We can regard $I^k(x,y)$ as $I(x,y,\frac{k}{T})$ in the diffussin equation.

Then: $\frac{\partial}{\partial t} I(x,y,t) = t\,\Delta I(x,y,t)$ can be discretized.

$$\frac{I^{k+1}(x,y) - I^{k}(x,y)}{\left(\frac{1}{T}\right)} = \frac{k}{T}\, p * I^{k}(x,y) \qquad (\star)$$

Given I^0, we get I^1 from $(\star)$

〃 I^1, 〃 〃 I^2 from $(\star)$

⋮

For Anisotropic diffusion, consider:

$$\frac{\partial}{\partial t} I(x,y,t) = t\ \nabla \cdot \left(k(x,y)\, I\,(x,y,t) \right)$$

It can be discretized:

$$\frac{I^{k+1}(x,y) - I^{k}(x,y)}{\left(\frac{1}{T}\right)} = \frac{k}{T}\, \nabla \cdot \left(k(x,y)\, I(x,y,t) \right) \quad (\ast)$$

Given I^0, we get I^1 from $(\ast)$

" I^1, " " I^2 from $(\ast)$

⋮

$K(x,y)$ can be choosen as: $K(x,y) = \dfrac{1}{|\nabla I(x,y)|}$ for edge-preserving image denoising.

Image denoising using energy minimization

Let g be a noisy image corrupted by additive noise n.

Then: $g(x,y) = \underbrace{f(x,y)}_{\text{Clean image}} + \underbrace{n(x,y)}_{\text{noise}}$

Recall: Laplacian masking : $g = f - \Delta f$ (Obtain a (non-smooth) sharp image from a smooth image)

Conversely, to get a smooth image f from a non-smooth image g, we can solve the PDE for f : $\underbrace{-\Delta f + f}_{\text{unknown}} = \underbrace{g}_{\text{known}}$

We will show that solving the above equation is equivalent to minimizing something :

$$E(f) = \iint \left(f(x,y) - g(x,y)\right)^2 dx\,dy + \iint \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx\,dy$$

In the discrete case, the PDE can be approximated (discretized) to get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)]$$

for all (x,y) (Linear System)

Solved by

Direct method (Big linear system)

Iterative method

Consider : $E_{discrete}(f) = \sum_{x=1}^{N} \sum_{y=1}^{N} (f(x,y) - g(x,y))^2 + \sum_{x=1}^{N} \sum_{y=1}^{N} [(f(x+1,y) - f(x,y))^2 + (f(x,y+1) - f(x,y))^2$

Suppose f is a minimizer of $E_{discrete}$. Then, for each (x,y),

$E_{discrete}$ depends on $f(x,y)$ for each (x,y)

$$\frac{\partial E_{discrete}}{\partial f(x,y)} = 0.$$

$$= 2(f(x,y) - g(x,y)) + 2(f(x+1,y) - f(x,y))(-1) + 2(f(x,y+1) - f(x,y))(-1)$$
$$+ 2(f(x,y) - f(x-1,y)) + 2(f(x,y) - f(x,y-1))$$

By simplification, we get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)]$$

The continuous version of $E_{discrete}$ can be written as:

$$E(f) = \iint (f(x,y) - g(x,y))^2 + \iint \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]}_{|\nabla f|^2} dx\,dy$$

Remark:

- Solving $f = g + \Delta f$ is equivalent to energy minimization
- The first term in Ediscrete is called the fidelity term. Aim to find f that is close to g.
- The second term is called the regularization term. Aim to enhance Smoothness.
- $-\Delta f + f = g$ can also be solved in the frequency domain:

$$DFT(f) = DFT(g + \underbrace{\Delta f}_{p * f})$$

$$\therefore DFT(f)(u,v) = DFT(g)(u,v) + c\, DFT(p)(u,v)\, DFT(f)(u,v)$$

$$\Longleftrightarrow DFT(f)(u,v) = \left[\frac{1}{1 - c\, DFT(p)(u,v)}\right] DFT(g)(u,v)$$

$\downarrow$ inverse DFT

$f(x,y)$!!

Image processing by minimization

(Variational approach)

1. Consider a minimization model (usually in continuous sense)
2. Derive a PDE related to minimization model
3. Discretize the PDE to get a linear system.

e.g. Total-variation (TV) denoising model. (ROF)
(Rudin - Osher - Fatemi)

2D integration by part formula

Let $f : [a,b] \times [a,b] \to \mathbb{R}$ and $g : [a,b] \times [a,b] \to \mathbb{R}$.

Assume $f(a,y) = f(b,y) = f(x,a) = f(x,b) = 0.$

$g(a,y) = g(b,y) = g(x,a) = g(x,b) = 0.$

Then: $$\int_a^b \int_a^b \nabla f(x,y) \cdot \nabla g(x,y)\, dx\, dy = -\int_a^b \int_a^b \Delta f(x,y)\, g(x,y)\, dx\, dy$$

Proof: $$\int_a^b \int_a^b \underbrace{\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}}_{\nabla f \cdot \nabla g} dx\, dy = -\int_a^b \int_a^b \left(\frac{\partial^2 f}{\partial x^2}\right) g\, dx\, dy + \int_a^b \cancelto{0}{\left(\frac{\partial f}{\partial x}\right) g \Big|_{x=a}^{x=b}} dy$$

$$-\int_a^b \int_a^b \left(\frac{\partial^2 f}{\partial y^2}\right) g\, dx\, dy + \int_a^b \cancelto{0}{\frac{\partial f}{\partial y}\, g \Big|_{y=a}^{y=b}} dx$$

$$= -\int_a^b \int_a^b \underbrace{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)}_{\Delta f} g\, dx\, dy$$

Also,

$$\int_a^b\int_a^b \left(k(x,y)\,\nabla f(x,y)\right)\cdot \nabla g(x,y)\,dx\,dy = -\int_a^b\int_a^b \nabla\cdot\left(k(x,y)\,\nabla f(x,y)\right) g(x,y)\,dx\,dy$$

where $k : [a,b]\times[a,b] \to \mathbb{R}$.

<u>Proof:</u>

$$\int_a^b\int_a^b \left[k(x,y)\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + k(x,y)\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right] dx\,dy$$

$$= -\int_a^b\int_a^b \frac{\partial}{\partial x}\left(k(x,y)\frac{\partial f}{\partial x}\right) g\,dx\,dy + \int_a^b \cancelto{0}{k(x,y)\frac{\partial f}{\partial x}\,g\Big|_{x=a}^{x=b}}\,dy$$

$$-\int_a^b\int_a^b \frac{\partial}{\partial y}\left(k(x,y)\frac{\partial f}{\partial y}\right) g\,dx\,dy + \int_a^b \cancelto{0}{k(x,y)\frac{\partial f}{\partial y}\,g\Big|_{y=a}^{y=b}}\,dx$$

$$= -\int_a^b\int_a^b \underbrace{\left[\frac{\partial}{\partial x}\left(k(x,y)\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(x,y)\frac{\partial f}{\partial y}\right)\right]}_{\nabla\cdot\left(k(x,y)\,\nabla f\right)} g\,dx\,dy$$

In general, we have:

<u>Useful Tool:</u> (Integration by part)

$$\int_\Omega \nabla f \cdot \nabla g \, dx\, dy = -\int_\Omega \big(\overbrace{\nabla \cdot}^{\text{divergence}} (\nabla f)\big)\, g \, dx\, dy + \int_{\partial\Omega} g \, (\nabla f \cdot \vec{n}) \, ds$$

$$\nabla \cdot \big(V_1(x,y), V_2(x,y)\big) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y}$$

where $\vec{n} = (n_1, n_2)$ = outward normal on the boundary.

or more generally,

$$\int_\Omega K(x,y)\, \nabla f(x,y) \cdot \nabla g(x,y) \, dx\, dy = -\int_\Omega \nabla \cdot \big(K(x,y)\, \nabla f(x,y)\big)\, g(x,y)\, dx\, dy + \int_{\partial\Omega} g(x,y) \big(K(x,y)\, \nabla f(x,y) \cdot \vec{n}\big)\, ds$$

Another useful fact:

If: $\int_\Omega T(x,y)\, v(x,y)\, dx\, dy = 0$ for all $v(x,y)$

then, we can conclude $T(x,y) = 0$ in Ω

Example: Suppose we have the following integral equation:

$$\int_a^b \int_a^b \big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\, dy + \int_a^b \int_a^b \nabla \cdot \nabla f(x,y)\, k(x,y)\, v(x,y)\, dx\, dy = 0$$

for all $v(x,y)$. Then: we have:

$$\int_a^b \int_a^b \Big[\big(f(x,y) - g(x,y)\big) + k(x,y)\, \nabla \cdot \nabla f(x,y)\Big] v(x,y)\, dx\, dy = 0$$

for all $v(x,y)$

We can conclude: $\big(f(x,y) - g(x,y)\big) + k(x,y)\, \nabla \cdot \nabla f(x,y) = 0$

for all $(x,y) \in [a,b] \times [a,b]$

Image denoising by solving PDE (derived from energy minimisation problem)

Consider the harmonic - L2 minimization model:

$$\text{minimize} \quad E(f) = \underbrace{\int_a^b \int_a^b}_{\text{image domain}} (f(x,y) - \underbrace{g(x,y)}_{\text{Observed}})^2 \, dx\, dy + \int_a^b \int_a^b \underbrace{|\nabla f|^2}_{\text{Smoothness of } f} \, dx\, dy$$

(Look for (continuous) image f)

Assume that $f(x,y) = g(x,y) = 0$ on the boundary of $[a,b] \times [a,b]$.

Suppose f minimizes $E(f)$. Let $v : [a,b] \times [a,b] \to \mathbb{R}$ such that $v(x,y) = 0$ on the boundary of $[a,b] \times [a,b]$.

Consider $f^{\varepsilon} = f + \varepsilon v : [a,b] \times [a,b] \to \mathbb{R}$, which is another image with $f^{\varepsilon}(x,y) = 0$ on the boundary of $[a,b] \times [a,b]$.

$$f^{\varepsilon}(x,y) = \underset{0}{\underset{\shortparallel}{f(x,y)}} + \varepsilon \underset{0}{\underset{\shortparallel}{v(x,y)}} = 0 \quad \text{on } \partial\big([a,b] \times [a,b]\big).$$

Consider $S : \mathbb{R} \to \mathbb{R}$ defined by:

$$S(\varepsilon) \stackrel{def}{=} E(f^{\varepsilon}) = E(f + \varepsilon v).$$

Note that $S(0) = E(f) =$ minimum of E. Thus, S attains its minimum at $\varepsilon = 0$.

$$\therefore \quad \frac{dS}{d\varepsilon}(0) = 0.$$

Now, $$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S(\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} E(f+\varepsilon v) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\left(\int_a^b\int_a^b (f(x,y)+\varepsilon v(x,y)-g(x,y))^2\,dx\,dy + \int_a^b\int_a^b |\nabla(f+\varepsilon v)(x,y)|^2\,dx\,dy\right)$$

$$|\nabla(f+\varepsilon v)(x,y)|^2 = (\nabla f + \varepsilon \nabla v)\cdot(\nabla f + \varepsilon \nabla v) = \nabla f\cdot\nabla f + 2\varepsilon \nabla f\cdot\nabla v + \varepsilon^2 \nabla v\cdot\nabla v = |\nabla f|^2 + 2\varepsilon \nabla f\cdot\nabla v + \varepsilon^2|\nabla v|^2$$

$$= \int_a^b\int_a^b 2(f(x,y)+\varepsilon v(x,y)-g(x,y))\,v(x,y)\Big|_{\varepsilon=0} dx\,dy$$

$$+ \int_a^b\int_a^b (2\nabla f\cdot\nabla v + 2\varepsilon|\nabla v|^2)\Big|_{\varepsilon=0} dx\,dy$$

$$= \int_a^b\int_a^b 2(f(x,y)-g(x,y))\,v(x,y)\,dx\,dy + \int_a^b\int_a^b 2\nabla f(x,y)\cdot\nabla v(x,y)\,dx\,dy$$

$$\therefore\ S'(0) = 0 = 2\int_a^b\int_a^b \big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\, dy + 2\int_a^b\int_a^b \left(\frac{\partial f}{\partial x}(x,y)\, \frac{\partial v}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\, \frac{\partial v}{\partial y}(x,y)\right) dx\, dy$$

for all $v(x,y)$. — $(*)$

If we can formulate $(*)$ in the form:

$$\int_a^b\int_a^b T(x,y)\, v(x,y) = 0 \quad \text{for all } v(x,y),$$

then we can conclude that $T(x,y) = 0$ in $[a,b]\times[a,b]$.

<u>Remark:</u>

- First term is in the form $\int_a^b\int_a^b T(x,y)\, v(x,y)$
- Second term is NOT.

Need to reformulate the second term.

Strategy: integration by part.

Second term: $$\int_a^b \int_a^b \nabla f(x,y) \nabla v(x,y)\, dx\, dy = -2\int_a^b \int_a^b \Delta f(x,y)\, v(x,y)\, dx\, dy.$$

All together, we have

$$0 = S'(0) = \int_a^b \int_a^b 2\big(f(x,y) - g(x,y)\big)\, v(x,y)\, dx\, dy - 2\int_a^b \int_a^b \Delta f(x,y)\, v(x,y)\, dx\, dy$$

$$\therefore \int_a^b \int_a^b \Big(2\big(f(x,y) - g(x,y)\big) - 2\,\Delta f(x,y)\Big)\, v(x,y)\, dx\, dy = 0 \text{ for all } v(x,y).$$

We conclude:

$$2\big(f(x,y) - g(x,y)\big) - 2\Delta f(x,y) = 0 \quad \text{for } (x,y) \in [a,b]\times[a,b]$$

or

$$f(x,y) - g(x,y) - \Delta f(x,y) = 0$$ (converse of Laplacian masking !!)