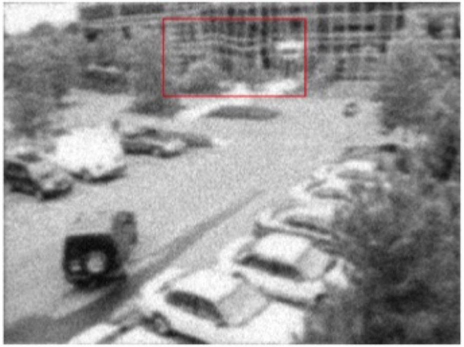
Lecture 9: Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

$$f \xrightarrow{D} g$$

D (not a matrix, just a transformation)

f: clean, original image

g: blurry image of f

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = D(f) + n$

where D is the degradation function/operator and n is the additive noise.

Assumption on D:

1. D is position invariant:

   Let $g(x,y) = D(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $D(\tilde{f})(x,y) = g(x-\alpha, y-\beta) = D(f)(x-\alpha, y-\beta)$

2. Linear: $D(f_1 + f_2) = D(f_1) + D(f_2)$

   $D(\alpha f) = \alpha D(f)$ where α is a scalar multiplication.

With the above assumption, we can show that: (assume indices taken between $-\frac{N}{2}$ to $\frac{N}{2}-1$)

$$D(f) = f * h \quad \text{where}$$

$$h = D(\underbrace{\delta}_{N \times N \text{ matrix}}) \qquad \delta(x,y) = \begin{cases} 1 & \text{if } (x,y)=(0,0) \\ 0 & \text{otherwise} \end{cases}$$

$\therefore$ With the above assumption,

Degradation / Blur = Convolution

Remark:

- $g(x,y) = h * f(x,y)$

In the frequency domain,

$$G(u,v) = \underbrace{c}_{\text{constant}}\, H(u,v)\, F(u,v)$$

$\therefore$ Deblurring can be done by:

Compute: $F(u,v) \approx \dfrac{G(u,v)}{cH(u,v)}$

$G(u,v)$ — from observe

$cH(u,v)$ — from known degradation

Obtain: $f(x,y) = DFT^{-1}(F(u,v))$

Image deblurring in the frequency domain: (Assume H is known)
(Assuming central spectrum)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{H(u,v) + \underbrace{\varepsilon\, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underbrace{G(u,v)}_{= H(u,v)F(u,v) + N(u,v)} T(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$$

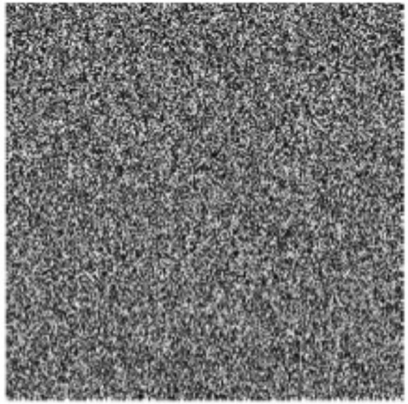
Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)
is small for (u,v) far away from $(0,0)$

$\therefore \dfrac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$

Large gain in high frequencies
$\Downarrow$
Boast up noises!!

Original

Blurred image



Direct inverse filtering

Method 2: Modified inverse filtering

Let $B(u,v) = \dfrac{1}{1 + \left(\dfrac{u^2+v^2}{D^2}\right)^n}$ and $T(u,v) = \dfrac{B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$.

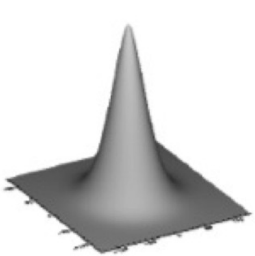
Then define: $\hat{F}(u,v) = T(u,v)\, G(u,v) \approx F(u,v)\, B(u,v) + \dfrac{N(u,v)\, B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$

$\dfrac{N(u,v)\, B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))} \approx \dfrac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ for $(u,v) \approx (0,0)$

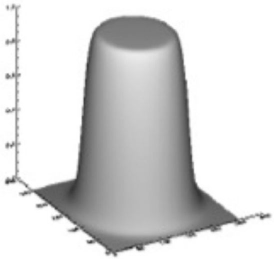
$\dfrac{N(u,v)\, B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is small (as $B(u,v)$ is small) for (u,v) far away from $(0,0)$.

$\dfrac{B(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ suppresses the high-frequency gain.

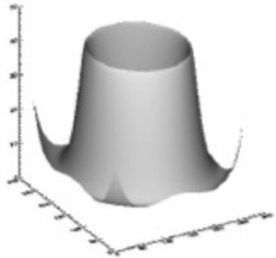
Bad: Has to choose D and n carefully.



$H(u,v)$



$B(u,v)$: $D = 90, n = 8$



Inverse B/H

Original Image $G(u,v)$

Blurred using $D = 90, n = 8$

Restored with a best D and n.

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y)$ = inverse DFT of $\hat{F}(u,v)$.

In fact, the Wiener filter can be described as an inverse filtering as follows:

$$\hat{F}(u,v) = \left[\left(\frac{1}{H(u,v)}\right)\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K}\right)\right] G(u,v)$$

Behave like "Modified inverse filtering"

≈ 0 if $H(u,v) \approx 0$ (if (u,v) far away from 0)

≈ 1 if $H(u,v)$ is large (if $(u,v) \approx (0,0)$)

What does Wiener filtering do mathematically?

We can show: Wiener filter minimizes the mean square error:

$$\underset{\substack{\uparrow \\ \text{original}}}{\varepsilon^2(f}, \underset{\substack{\uparrow \\ \text{Restored}}}{\hat{f}}) = \sum_{x=-\frac{N}{2}}^{\frac{N}{2}} \sum_{y=-\frac{N}{2}}^{\frac{N}{2}} \left| f(x,y) - \hat{f}(x,y) \right|^2$$

Let $\underset{\text{Observed}}{g} = \overset{\text{degradation}}{h} * \underset{\text{original}}{f} + \underset{\text{noise}}{n}$

Then, the restored image $\hat{f}(x,y)$ can be written as:

$$\hat{f}(x,y) = w(x,y) * g(x,y) \text{ for some } w(x,y)$$

Recall: $\hat{f}$ is obtained as follows:

Step 1: Let $\hat{F}(u,v) = \underbrace{W(u,v)}_{\text{Filter}} G(u,v)$ $\quad (G(u,v) = \mathrm{DFT}(g)(u,v))$

Step 2: Compute iFT of $\hat{F}$ to get $\hat{f}$

$\therefore\ \hat{f} = c\,w * g$ for some w. $\quad$ (or $\hat{F} = \mathrm{DFT}(\hat{f}) = W \odot G$)

Thus, $\hat{f}$ denpends on W

We can regard $\mathcal{E}^2(\hat{f}, f)$ as a functional depending on W:

$$\mathcal{E}^2(\hat{f}, f) = \mathcal{E}^2(W)$$

We can find an optimal W that minimizes:

$$\mathcal{E}^2(W) = \| \hat{f}(W) - f \|_F^2$$

Under suitable condition (spatially correlated), the minimizer W is given by:

$$W(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}} \quad \text{where} \quad \begin{aligned} S_n(u,v) &= |N(u,v)|^2 \\ S_f(u,v) &= |F(u,v)|^2 \end{aligned}$$

Method 4: Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = \underbrace{h}_{\text{degradation}} * f + \underbrace{n}_{\text{noise}}$

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$

$\vec{g} = \mathcal{S}(g)$, $\vec{f} = \mathcal{S}(f)$, $\vec{n} = \mathcal{S}(n)$

$\mathcal{S}(g)$: Stacked image of g

D: transformation matrix of $h*f$ (or f)

$\vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

Constrained least square problem:

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\Delta f(x,y)|^2 \quad \text{subject to the constraint:}$$

$$\|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

What is Δf ?

In the discrete case, we can estimate:

$$\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f}{\partial x^2}(x,y) \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

$$\xrightarrow{\text{Put } h=1} \quad \Delta f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

$\therefore$ Δ is the Laplacian in the discrete case

Remark:

- More generally, $\Delta f = p * f$ ← discrete convolution

  where

$$
\begin{array}{cl}
p = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ & & 1 & & \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} & \begin{matrix} \\ \\ x=0 \\ \\ \\ \end{matrix} \\
\qquad\qquad y=0 &
\end{array}
$$

- Minimizing $\displaystyle\sum_{x=0}^{N-1} \sum_{y=0}^{N-1} |\Delta f(x,y)|^2$ is to denoise.

- Also, $\vec{g} = D\vec{f} + \vec{n} \iff \vec{g} - D\vec{f} = \vec{n} \Rightarrow \|\vec{g} - D\vec{f}\|^2 = \|\vec{n}\|^2 \approx \varepsilon$ ← noise level

  $\therefore$ the constraint $\|\vec{g} - D\vec{f}\|^2$ is to solve the deblurring problem.

- $\displaystyle\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left| \Delta f(x,y) \right|^2$ $\longleftarrow$ Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ $\longleftarrow$ Deblur

<u>Remark:</u> $\underbrace{\|\vec{g} - D\vec{f}\|^2}_{=\,\|\vec{n}\|^2} = \varepsilon$ means we Allow some fixed level of noise.

Let $\overrightarrow{p*f} = \mathcal{S}(p*f) = L\vec{f}$ transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$\left(D^T D + \gamma L^T L\right)\vec{f} = D^T\vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma|P(u,v)|^2}G(u,v)$$

$\Big(H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}\Big)$$