Lecture 8 Recall:

Image enhancement in the frequency domain:

Goal: 1. Remove $\underbrace{\text{high-frequency components}}_{\text{noise}}$ (low-pass filter) for image denoising.

2. Remove $\underbrace{\text{low-frequency components}}_{\text{non-edge}}$ (high-pass filter) for the extraction of image details.

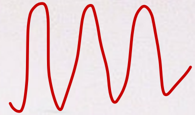
Let $\hat{F}$ be the DFT of an $N \times N$ image F. (indices taken from 0 to $N-1$)

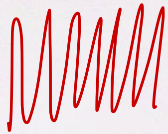
Then: for all $0 \leq m, n \leq N-1$,

$$\hat{\hat{F}}(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{F}(k,l)\, e^{j\frac{2\pi}{N}(km+ln)}$$

$\therefore$ $\hat{F}(k,l)$ is associated to the complex function $g(m,n) = e^{j\frac{2\pi}{N}(km+ln)}$

Goal: Remove "jumpy" components by setting suitable $\hat{F}(k,l)$ to zero.

 = a  + b 

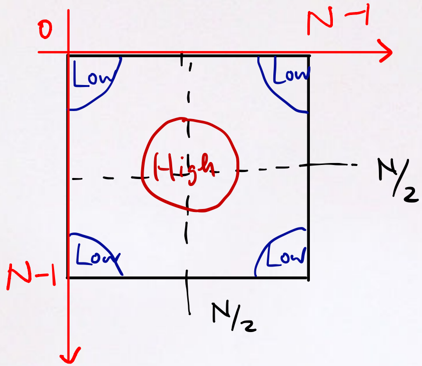
+ c 

To remove noise, truncate c (let $c = 0$)

Observation:

If the image I takes indices between 0 to N-1,
then the DFT of I takes indices between 0 to N-1.

We have:

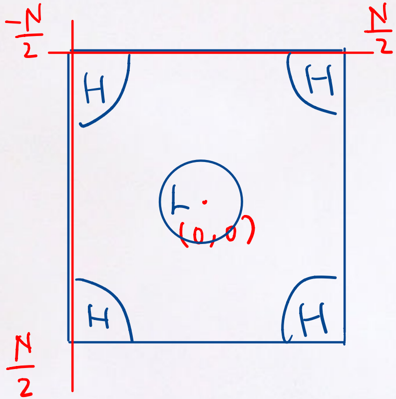


Centralisation:

Let F be an image whose indices are taken between $-\frac{N}{2}$ to $\frac{N}{2}$

Then, $DFT(F)$ is a matrix whose indices are also taken between $-\frac{N}{2}$ to $\frac{N}{2}$.

We have:



Procedures for image processing by modifying Fourier coefficients

Given an image $I = (I_{ij})_{-\frac{N}{2} \le i,j \le \frac{N}{2}}$.

1. Compute DFT of I (Denote $\hat{I} = DFT(I)$)

2. Then: obtain a new DFT matrix, $\hat{I}^{new}$, by:

$$\hat{I}^{new} = H \odot \hat{I} \qquad \left(\text{Here } H \underset{\substack{\text{pixel-wise}\\ \text{multiplication}}}{\odot} \hat{I}(u,v) = H(u,v)\hat{I}(u,v)\right)$$

H is a suitable filter.

3. Finally, obtain an improved image by inverse DFT:

$$I^{new} = \underbrace{iDFT}_{\text{inverse DFT}}(\hat{I}^{new})$$

Note: Let $h = \underbrace{\text{iDFT}}_{\text{inverse DFT}}(H)$

$$H \odot \hat{I} \xrightarrow{\text{inverse DFT}} \underset{\substack{\uparrow \\ \text{normalizing} \\ \text{constant}}}{C}\ h * I$$

Example of Low-pass filters for image denoising

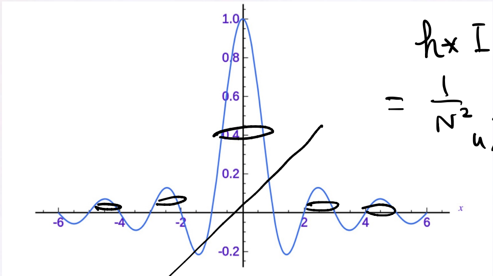
Assume that we work on the centered spectrum!

That is, consider $\hat{F}(u,v)$ where $-N/2 \le u \le N/2 - 1$, $-N/2 \le v \le N/2 - 1$.

1 Ideal low pass filter (ILPF):

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) := u^2 + v^2 \le D_0^2 \\ 0 & \text{if } D(u,v) > D_0^2 \end{cases}$$

In 1-dim cross-section, iDFT$(H(u,v))$ looks like:



$h * I(x,y)$

$= \frac{1}{N^2} \sum_{u,v} h(x-u, y-v)\, I(u,v)$

every pixel values of I has an effect on $h * I(x,y)$!!

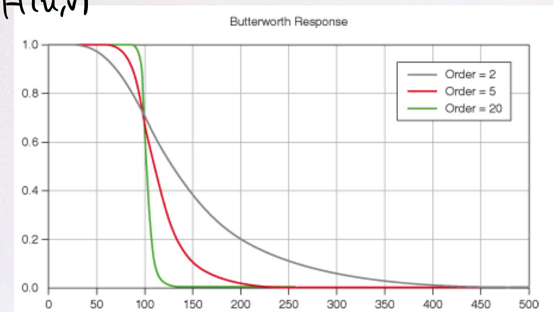
Good: Simple

Bad : Produce ringing effect!

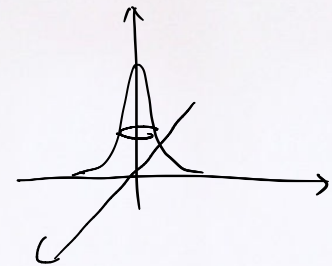
2. Butterworth low-pass filter (BLPF) of order n ($n \geq 1$ integer):

$$H(u,v) = \frac{1}{1 + \left(D(u,v)/D_0\right)^{n}}$$

$H(u,v)$ in 1-dim



$\mathcal{F}^{-1}(H(u,v))$ in 1-dim



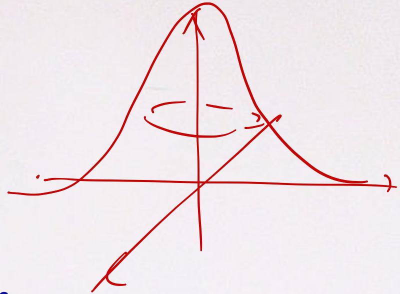
Good: Produce less / no visible ringing effect if n is carefully chosen!!

3. Gaussian low-pass filter

$u^2 + v^2$

$$H(u,v) = \exp\left(-\frac{D(u,v)}{2\sigma^2}\right)$$

σ = spread of the Gaussian function



F. T. of Gaussian is also Gaussian!!

Good: No visible ringing effect!!

Examples for high-pass filtering for feature extraction

1. Ideal high-pass filter: (IHPF)

$$H(u,v) = \begin{cases} 0 & \text{if} \quad D(u,v) \le D_0^2 \\ 1 & \text{if} \quad D(u,v) > D_0^2 \end{cases}$$

Bad: Produce ringing

2. Butterworth high-pass filter:

$$H(u,v) = \frac{1}{1 + \left(\frac{D_0}{D(u,v)}\right)^n}$$

$\left(H(u,v) = 0 \text{ if } D(u,v) = 0 \right)$

Choose the right n

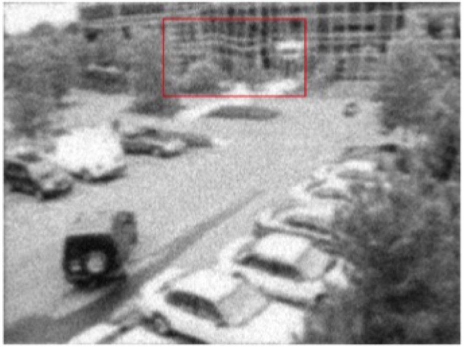
Good: Less ringing

3. Gaussian high-pass filter

$$H(u,v) = 1 - e^{-\left(\frac{D(u,v)}{2\sigma^2}\right)}$$

Good: No visible ringing!

Image deblurring



Atmospheric turbulence

Motion Blur

Speeding problem

Image deblurring in the frequency domain:

Mathematical formulation of image blurring

$$f \xrightarrow{D} g$$

D (not a matrix, just a transformation)

f: clean, original image

g: blurry image of f

Let g be the observed (blurry) image.

Let f be the original (good) image.

Model g as: $g = D(f) + n$

where D is the degradation function/operator and n is the additive noise.

Assumption on D:

1. D is position invariant:

   Let $g(x,y) = D(f)(x,y)$ and let $\tilde{f}(x,y) := f(x-\alpha, y-\beta)$.

   Then: $D(\tilde{f})(x,y) = g(x-\alpha, y-\beta) = D(f)(x-\alpha, y-\beta)$

2. Linear: $D(f_1 + f_2) = D(f_1) + D(f_2)$

   $D(\alpha f) = \alpha D(f)$ where α is a scalar multiplication.

<u>Claim:</u> With the above assumption, we can show that: (assume indices taken between $-\frac{N}{2}$ to $\frac{N}{2}-1$)

$$D(f) = f * h \quad \text{where}$$

$$h = D(\underbrace{\delta}_{N \times N \text{ matrix}}) \qquad \delta(x,y) = \begin{cases} 1 & \text{if } (x,y)=(0,0) \\ 0 & \text{otherwise} \end{cases}$$

With the above assumption, consider an impluse image $\delta \in M_{(N+1)\times(N+1)}$ (indices taken between $-\frac{N}{2}$ to $\frac{N}{2}$)

$$\delta(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 0 & \text{if } (x,y) \neq (0,0) \end{cases}$$

Let $\tilde{\delta}_{\alpha,\beta}$ be the translated image of δ by (α,β):

$$\tilde{\delta}_{\alpha,\beta}(x,y) = \delta(x-\alpha, y-\beta) \quad \text{for} \quad -\frac{N}{2} \leq x, y \leq \frac{N}{2}$$

Note: $f(x,y) = f * \delta(x,y) = \displaystyle\sum_{\alpha=-\frac{N}{2}}^{N/2-1} \sum_{\beta=-N/2}^{N/2-1} f(\alpha,\beta)\,\delta(x-\alpha, y-\beta) = \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\,\tilde{\delta}_{\alpha,\beta}(x,y)$

for all $-\frac{N}{2} \leq x, y \leq \frac{N}{2}$.

$$\therefore \quad f = \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} \underbrace{f(\alpha,\beta)}_{\in\, \mathbb{R}} \underbrace{\tilde{\delta}_{\alpha,\beta}}_{M_{(N+1)\times(N+1)}}$$

Let g be the blurry image of f. That is, $g = D(f)$

$$g = D(f) = D\left(\sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, \tilde{\delta}_{\alpha,\beta}\right)$$

$$= \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, D(\tilde{\delta}_{\alpha,\beta}) \quad \text{(linearity of } D\text{)}$$

$$\therefore\ g(x,y) = \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, D(\tilde{\delta}_{\alpha,\beta})(x,y)$$

$$= \sum_{\alpha=-\frac{N}{2}}^{\frac{N}{2}} \sum_{\beta=-\frac{N}{2}}^{\frac{N}{2}} f(\alpha,\beta)\, D(\delta)(x-\alpha, y-\beta) \quad \text{(position-invariant)}$$

$$= f * h(x,y) \quad \text{where} \quad h = D(\delta)$$

$$\therefore\ g = f * h$$

$\therefore$ With the above assumption,

$$\text{Degradation/Blur} = \text{Convolution}$$

Remark:

- $g(x,y) = h * f(x,y)$

In the frequency domain,

$$G(u,v) = \underset{\uparrow\ \text{constant}}{c}\, H(u,v)\, F(u,v)$$

$\therefore$ Deblurring can be done by:

Compute: $F(u,v) \approx \dfrac{G(u,v)}{cH(u,v)}$ — from observe. — from known degradation

$\Downarrow$

Obtain: $f(x,y) = DFT^{-1}(F(u,v))$

Examples of degradation function $H(u,v)$

1. Atmospheric turbulence blur:

$$H(u,v) = e^{-k(u^2+v^2)^{5/6}}$$

where k = degree of turbulence

$k = 0.0025$ (severe)

$k = 0.001$ (mild)

$k = 0.00025$ (low turbulence)

2. Out of focus blur:

In the frequency domain, define $H(u,v)$ as the DFT of

$$h(x,y) = \begin{cases} 1 & \text{if } x^2+y^2 \le D_0^2 \\ 0 & \text{otherwise} \end{cases}$$

$$H \odot \hat{I}$$

$$\updownarrow$$

$$h * I$$

3. Uniform Linear Motion Blur

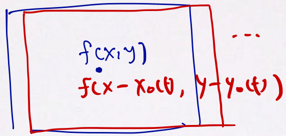
Assume $f(x,y)$ undergoes planar motion during acquisition.
(original)

(displacements)

Let $(x_0(t), y_0(t))$ be the motion components in the x- and y-directions

time

Let T be the total exposure time.



The observed image is given by:

$$g(x,y) = \int_0^T f(x - x_0(t), y - y_0(t))\, dt$$

Now, let $G(u,v) = \mathrm{DFT}(g)(u,v)$, then:

$$G(u,v) = \frac{1}{N^2} \sum_x \sum_y g(x,y)\, e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \frac{1}{N^2} \sum_x \sum_y \int_0^T f(x - x_0(t), y - y_0(t))\, dt\; e^{-j\frac{2\pi}{N}(ux+vy)}$$

$$= \int_0^T \left(\sum_x \sum_y f(x - x_0(t), y - y_0(t))\, e^{-j\frac{2\pi}{N}(ux+vy)} \right) dt$$

Recall that $DFT(f(x-x_0, y-y_0)) = F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}$

$F = DFT(f)$

We have: $G(u,v) = \int_0^T \left[F(u,v)\, e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))} \right] dt$

$$= F(u,v) \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

$$= F(u,v)\, H(u,v)$$

$\therefore$ Degradation function in the frequency domain is given by:

$$H(u,v) = \int_0^T e^{-j\frac{2\pi}{N}(u x_0(t) + v y_0(t))}\, dt$$

(Speeding problem !!)

Example: Suppose the camera is moving left horizontally with a constant speed c.

That is, the image at time t is given by:

$$I^*(x, y) = I(x, y - ct)$$

Then: the degradation function is given by:

$$H(u, v) = \int_0^T e^{-j\frac{2\pi}{N}(v(ct))} dt$$

Remark: Once the degradation function is known, the original image can be restored by: $\mathrm{IDFT}\left(\frac{G(u,v)}{H(u,v)}\right)$ (given that there's no noise)

What if there is noise??

Image deblurring in the frequency domain: (Assume H is known)

Method 1: Direct inverse filtering

Let $T(u,v) = \dfrac{1}{H(u,v) + \underbrace{\varepsilon\, \mathrm{sgn}(H(u,v))}_{\text{Avoid singularity}}}$ ($\mathrm{sgn}(z) = 1$ if $\mathrm{Re}(z) \geq 0$ and $\mathrm{sgn}(z) = -1$ otherwise)

Compute $\hat{F}(u,v) = G(u,v)\, T(u,v)$.

Find inverse DFT of $\hat{F}(u,v)$ to get an image $\hat{f}(x,y)$.

Good: Simple

Bad: Boast up noise

$$\hat{F}(u,v) = \underbrace{G(u,v)}_{H(u,v)F(u,v) + N(u,v)}\, T(u,v) \approx F(u,v) + \frac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$$

Note: $H(u,v)$ is big for (u,v) close to $(0,0)$ (Keep low frequencies)
is small for (u,v) far away from $(0,0)$

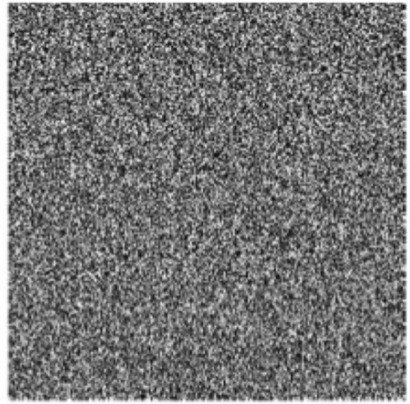
$\therefore \dfrac{N(u,v)}{H(u,v) + \varepsilon\, \mathrm{sgn}(H(u,v))}$ is big for (u,v) far away from $(0,0)$ — Large gain in high frequencies

$\Downarrow$

Boast up noises!!

Original

Blurred image



Direct inverse filtering