Lecture 6:

Recall:

Discrete Fourier Transform:

Definition:

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \underbrace{\hat{g}(m,n)}_{\text{DFT of } g}\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

(no $\frac{1}{Mn}$!) (no -ve sign)

DFT in Matrix form

Theorem: Consider a $N \times N$ image g, the DFT of g can be written as:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (U_{k\ell})_{0 \le k, \ell \le N-1} \in M_{N \times N}$ and $\mathcal{U}_{k\ell} = \frac{1}{N} e^{-j\frac{2\pi k\ell}{N}}$.

Theorem: $\mathcal{U}^* \mathcal{U} = \frac{1}{N} I$ where $\mathcal{U}^* = (\overline{\mathcal{U}})^T$ (conjugate transpose)

$\mathcal{U}\mathcal{U}^* = \frac{1}{N} I.$

$(\overline{a+jb} = a - jb)$

$(\overline{e^{j\theta}} = \overline{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
$= e^{-j\theta})$

$\therefore \mathcal{U}^{-1} = (N\mathcal{U})^*$

Theorem: $U^* U = \frac{1}{N} I$ where $U^* = (\overline{U})^T$ (conjugate transpose)

$(\overline{a+jb} = a - jb)$

$(e^{j\theta} = \overline{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta$
$= e^{-j\theta})$

$U U^* = \frac{1}{N} I.$

$\therefore U^{-1} = (NU)^*$

Proof: Consider $(U^* U)(k,l)$ (k-th row, l-th col of U^*U)

Let $U = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}$

$(U^* U)(k,l) = (\underline{\quad\text{k-th row of } U^*\quad}) \begin{pmatrix} | \\ | \end{pmatrix}$ l-col of U

$$= \overline{(\vec{u}_k^T)}\, \vec{u}_l$$

$$= \left(\overline{\frac{e^{-j2\pi\frac{k(0)}{N}}}{N}}, \ldots, \overline{\frac{e^{-j2\pi\frac{k\alpha}{N}}}{N}}, \ldots, \overline{\frac{e^{-j2\pi\frac{k(N-1)}{N}}}{N}} \right) \begin{pmatrix} \frac{e^{-j2\pi\frac{l(0)}{N}}}{N} \\ \vdots \\ \frac{e^{-j2\pi\frac{l(\alpha)}{N}}}{N} \\ \vdots \\ \frac{e^{-j2\pi\frac{l(N-1)}{N}}}{N} \end{pmatrix}$$

$$= \sum_{\alpha=0}^{N-1} \frac{e^{j2\pi\frac{k\alpha}{N}}}{N} \frac{e^{-j2\pi\frac{l\alpha}{N}}}{N} = \sum_{\alpha=0}^{N-1} \frac{e^{-j2\pi(\alpha)(l-k)}}{N^2}$$

$$= \frac{1}{N}\delta(l-k)$$

$$\therefore \quad \mathcal{U}^*\mathcal{U}(k,l) = \begin{cases} \frac{1}{N} & \text{if } k=l \\ 0 & \text{if } k \neq l \end{cases}$$

$$\Rightarrow \quad \mathcal{U}^*\mathcal{U} = \frac{1}{N} I$$

Similarly, $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I$

Image decomposition by DFT

Suppose $\hat{g} = \mathrm{DFT}(g) = \mathcal{U} g \mathcal{U}$

Then: $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^{\,T}$ ← Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$

Remark:

Note that $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I$. $\therefore$ $\mathcal{U}$ is not unitary.

If we normalize $\mathcal{U}$ to $\tilde{\mathcal{U}} = \sqrt{N}\,\mathcal{U}$. Then $\tilde{\mathcal{U}}$ is unitary!

Some other definition of DFT:

$$(2D) \quad \hat{f}(m,n) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} f(k,l)\, e^{-j2\pi\left(\frac{mk+nl}{N}\right)}$$

In this case, let $\tilde{\mathcal{U}} = (\tilde{\mathcal{U}}_{kl})_{0 \le k,l \le N-1}$; $\tilde{\mathcal{U}}_{kl} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi kl}{N}}$. Then:

$\hat{f} = \tilde{\mathcal{U}} f \tilde{\mathcal{U}}$

Then, $\tilde{\mathcal{U}} = \sqrt{N}\,\mathcal{U}$

$\therefore$ Normalizing the definition of DFT $\Rightarrow$ unitary $\tilde{\mathcal{U}}$ can be applied!

BUT: Inverse DFT must be adjusted!!

Mathematics of JPEG (Optional)

Consider a $M \times N$ image f. Extend f to a $2M \times 2N$ image $\tilde{f}$, whose indices are taken from $[-M, M-1]$ and $[-N, N-1]$.

Define $f(k,l)$ for $-M \leq k \leq M-1$ and $-N \leq l \leq N-1$ such that

$$f(-k-1, -l-1) = f(k,l) \quad \Big\} \text{ Reflection about } (-1/2, -1/2)$$

$$\left.\begin{aligned} f(-k-1, l) &= f(k,l) \\ f(k, -l-1) &= f(k,l) \end{aligned}\right\} \text{ Reflection about the axis } k = -1/2 \text{ and } l = -1/2$$

Example:

$l = -1/2$

$k = -1/2$

$$\begin{array}{cccccc|c} 9 & 8 & 7 & 7 & 8 & 9 & k=-3 \\ 6 & 5 & 4 & 4 & 5 & 6 & k=-2 \\ 3 & 2 & 1 & 1 & 2 & 3 & k=-1 \\ 3 & 2 & 1 & 1 & 2 & 3 & k=0 \\ 6 & 5 & 4 & 4 & 5 & 6 & k=1 \\ 9 & 8 & 7 & 7 & 8 & 9 & k=2 \\ \hline l=-3 & l=-2 & l=-1 & l=0 & l=1 & l=2 & \end{array}$$

$f(-1, 1)$
$=$
$f(0, 1)$

Reflection about the axis $k = -1/2$.

$f(-2, -2)$
$=$
$f(1, 1)$

Reflection about $(-1/2, -1/2)$.

Make the extension as a reflection about $(0,0)$, the axis $k=0$ and the axis $l=0$.

Done by shifting the image by $(1/2, 1/2)$

After shifting

$$\begin{array}{cccccc|c}
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+(-3) \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+(-2) \\
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+(-1) \\
\hline
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+0 \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+1 \\
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+2 \\
\hline
\frac{1}{2}+-3 & \frac{1}{2}+-2 & \frac{1}{2}+-1 & \frac{1}{2}+0 & \frac{1}{2}+1 & \frac{1}{2}+2 &
\end{array}$$

k

l

Before shifting:

DFT of f, denoted by F is given by:

$$F(m,n) = \hat{f}(m,n) = \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

After shifting, the image becomes:

$$\tilde{f}\left(k+\frac{1}{2}, l+\frac{1}{2}\right) = f(k,l) \quad \text{where} \quad \begin{array}{l} -M \leq k \leq M-1 \\ -N \leq l \leq N-1 \end{array}$$

DFT F is given by:

$$F(m,n) = \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} \underbrace{\tilde{f}\left(k+\frac{1}{2}, l+\frac{1}{2}\right)}_{\overset{\shortparallel}{f(k,l)}} e^{-j2\pi\left(\frac{\left(k+\frac{1}{2}\right)m}{M} + \frac{\left(l+\frac{1}{2}\right)n}{N}\right)}$$

Now, we compute the DFT of (shifted) $\tilde{f}$:

$$F(m,n) = \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j\frac{2\pi}{2M}m(k+\frac{1}{2})} e^{-j\frac{2\pi}{2N}n(l+\frac{1}{2})}$$

$$= \frac{1}{4MN} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

$$= \frac{1}{4MN} (\underbrace{\sum_{k=-M}^{-1} \sum_{l=-N}^{-1}}_{A_1} + \underbrace{\sum_{k=-M}^{-1} \sum_{l=0}^{N-1}}_{A_2} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=-N}^{-1}}_{A_3} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=0}^{N-1}}_{A_4})$$

$$f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

After some messy simplication, we can get:

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

Definition: (Even symmetric discrete cosine transform [EDCT])

Let f be a $M \times N$ image, whose indices are taken as $0 \le k \le M-1$ and $0 \le l \le N-1$. The **even symmetric discrete cosine transform (EDCT)** of f is given by:

$$\hat{f}_{ec}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

with $0 \le m \le M-1, 0 \le n \le N-1$

Remark:

- Smart idea to get a decomposition consisting only of cosine function (by reflection and shifting!)
- Can be formulated in matrix form
- Again, it is a separable image transformation.

- The inverse of EDCT can be explicitly computed. More specifically, the **inverse EDCT** is defined as:

$$f(k,l) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{ec}(m,n)\cos\frac{\pi m(2k+1)}{2M}\cos\frac{\pi n(2l+1)}{2N} \qquad (**)$$

where $C(0) = 1, C(m) = C(n) = 2$ for $m,n \neq 0$

Also involving cosine functions only!

- Formula (**) can be expressed as matrix multiplication:

$$f = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \hat{f}_{ec}(m,n)\vec{T}_m \vec{T'}_n^T$$

elementary images under EDCT!

where: $\vec{T}_m = \begin{pmatrix} T_m(0) \\ T_m(1) \\ \vdots \\ T_m(M-1) \end{pmatrix}, \vec{T'}_n = \begin{pmatrix} T'_n(0) \\ T'_n(1) \\ \vdots \\ T'_n(N-1) \end{pmatrix}$ with $T_m(k) = C(m)\cos\frac{\pi m(2k+1)}{2M}$

and $T'_n(k) = C(n)\cos\frac{\pi n(2k+1)}{2N}$.

This is what JPEG does!!

Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $$g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$$

$$\left(g, m \in M_{N\times M}(\mathbb{R}) \right)$$

pixel-wise multiplication

Then, the DFT of $g * w$ $= MN\, \mathrm{DFT}(g) \odot \mathrm{DFT}(w)$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Proof:

DFT of $g*w$ at (p,q)

$$= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g(n-n', m-m')\, w(n',m')\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

Change of variables:

$n \to n'' = n - n'$

$m \to m'' = m - m'$

$$= \underbrace{\frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} w(n',m')\, e^{-j2\pi\left(\frac{pn'}{N} + \frac{qm'}{M}\right)}}_{\hat{w}(p,q)} \underbrace{\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{M-1-m'} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that: g and w are periodically extended.

$\therefore\ g(n-N, m) = g(n,m)$ and $g(n, m-M) = g(n,m)$

$$\therefore\ T \equiv \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'',m'')\, e^{-j2\pi\frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi\frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\displaystyle\sum_{n''=-n'}^{-1} g(n'',m'')\, e^{-j2\pi \frac{pn''}{N}} \overset{n'''=N+n''}{=\!=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N,\ m'')}_{g(n''',\ m'')}\, e^{-j2\pi\left(\frac{pn''}{N}\right)} \cancelto{1}{e^{j2\pi p}}$

We can do similar thing for index m''.

$$\therefore\ T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'',m'')\, e^{-j2\pi\left(\frac{pn''}{N}+\frac{qm''}{M}\right)} = MN\,\hat{g}(p,q)$$

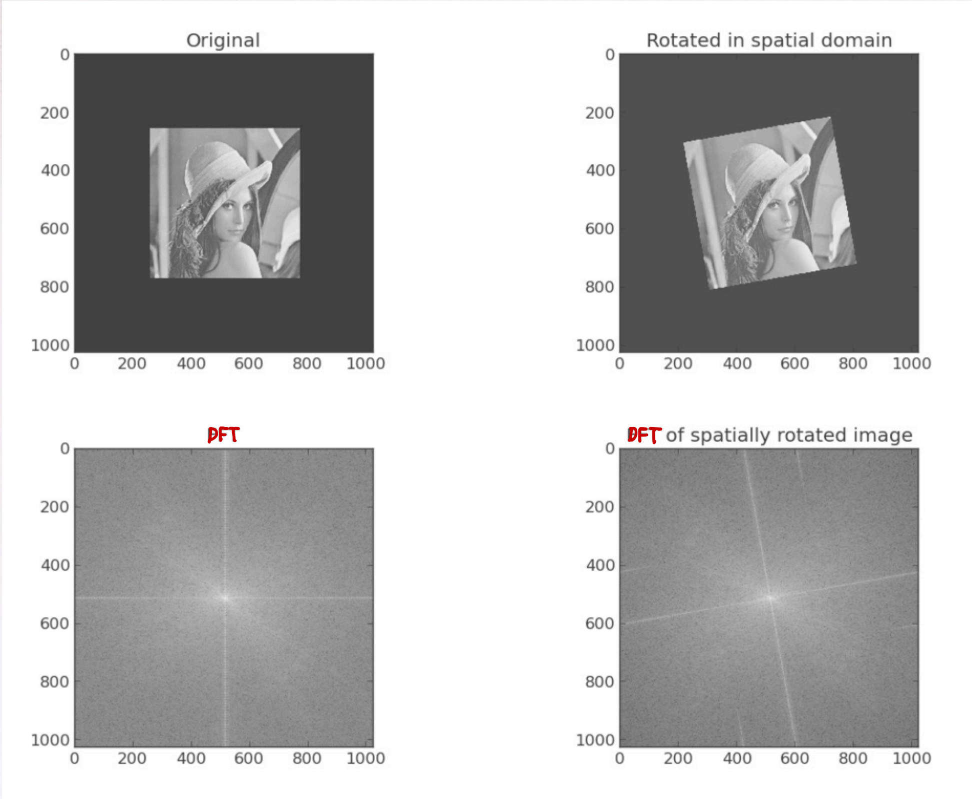
$$\therefore\ \widehat{g*w}(p,q) = MN\,\hat{g}(p,q)\,\hat{w}(p,q)$$

Note:

(Spatial domain) $I * g$ (Linear filtering: Linear combination of neighborhood pixel values)

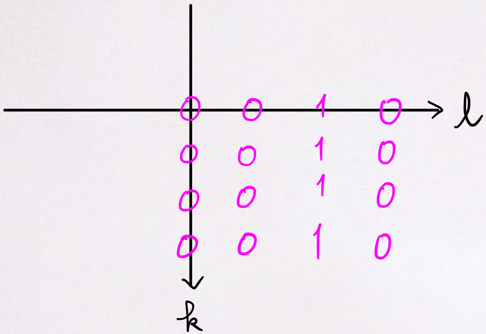
$$\Big\downarrow \ \text{DFT}$$

(Frequency domain) $MN\,\hat{I} \underbrace{\odot}_{\text{pixel-wise multiplication}} \hat{g}$ (Modifying the Fourier coefficients by multiplication)


Original
Rotated in spatial domain
DFT
DFT of spatially rotated image
0 200 400 600 800 1000

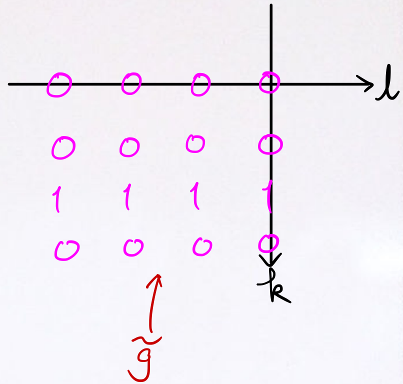
Example: Let $g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then: $\hat{g} = \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Note that g in the coordinate system:



Rotated by 90°

clockwisely



Note that indices of $\tilde{g}$ are taken as: $\begin{cases} -3 \le l \le 0 \\ 0 \le k \le 3 \end{cases}$.

Now, DFT of $\tilde{g} = \hat{\tilde{g}}$ (given by: $\displaystyle\sum_{k=0}^{3}\sum_{l=-3}^{0} \tilde{g}(k,l)\, e^{-j2\pi\left(\frac{km+ln}{4}\right)}$

$$= \begin{array}{c|c} \begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \end{pmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \; k \\ l \quad \begin{matrix} -3 & -2 & -1 & 0 \end{matrix} & \end{array}$$

$\because$ $0 \le k \le 3$

$-3 \le l \le 0$