Lecture 5 Recall

Separable image transformation $\mathcal{O}$

Given image f, transform f by $\mathcal{O}$ to get g (coefficient matrix)

$$g = \mathcal{O}(f) = A f B$$

Let $\tilde{A} = A^{-1}$ and $\tilde{B} = B^{-1}$. Then:

$$f = A^{-1} g B^{-1} = \tilde{A} g \tilde{B}$$

$$\tilde{A} = \begin{pmatrix} | & | & & | \\ \tilde{a}_1 & \tilde{a}_2 & \cdots & \tilde{a}_N \\ | & | & & | \end{pmatrix}$$

$$\tilde{B} = \begin{pmatrix} - & \vec{\tilde{b}}_1 & - \\ - & \tilde{b}_2 & - \\ & \vdots & \\ - & \tilde{b}_N & - \end{pmatrix}$$

Important formula:

$$f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \boxed{\begin{pmatrix} | \\ \tilde{a}_i \\ | \end{pmatrix} \begin{pmatrix} - & \tilde{b}_j & - \end{pmatrix}}$$

$N \times N$ elementary image

(Remove terms with)
Take away small g_{ij} ← Image compression.

Recall

Haar transformation

(From now on, we assume all images $f = (f_{ij})_{\substack{0 \le i \le N-1 \\ 0 \le j \le N-1}}$

Definition: (Haar functions) The Haar functions are defined as follows

$$H_0(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

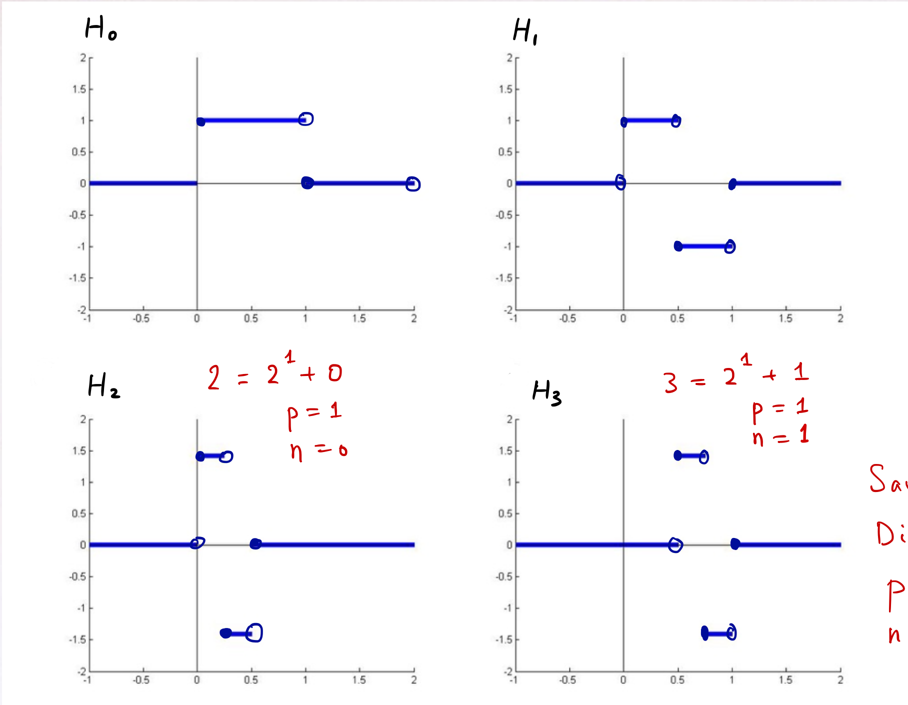
$$H_1(t) \equiv \begin{cases} 1 & \text{if } 0 \le t < 1/2 \\ -1 & \text{if } 1/2 \le t < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$H_{2^p+n} \equiv \begin{cases} \sqrt{2}^{\,p} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -\sqrt{2}^{\,p} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{elsewhere} \end{cases}$$

where $p = 1, 2, \quad , \; n = 0, 1, 2, \quad , 2^p - 1$

Remark: If p is larger, H_{2^p+n} is compactly supported region

Examples of Haar functions:



Same wavefront

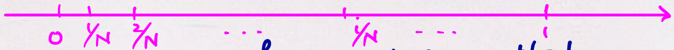
Different locations

$p \leftrightarrow$ wavefront

$n \leftrightarrow$ location

Definition (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is done by dividing $[0,1]$ into partitions



Let $H(k,i) \equiv H_k\left(\frac{i}{N}\right)$ where $k, i = 0, 1, 2, \ldots, N-1$

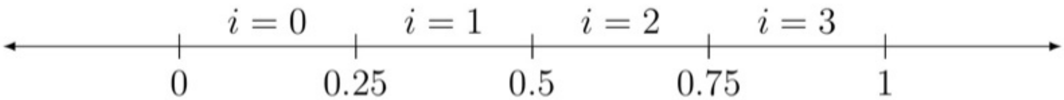
We obtain the Haar Transform matrix $\tilde{H} \equiv \frac{1}{\sqrt{N}} H$ where $H \equiv \left(H(k,i)\right)_{0 \le k, i \le N-1}$

The Haar Transform of $f \in M_{n \times n}$ is defined as

$$g = \tilde{H} f \tilde{H}^T$$

Example Compute the Haar Transform matrix for a 4×4 image.

Solution: Divide $[0,1]$ into 4 portions:



Example 2 Compute the Haar Transform of

$$f = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Solution:

$$g = \tilde{H} f \tilde{H}^T = \underbrace{\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}}_{\text{More zeros}}$$

Example 3 Suppose g in Example 2 is changed to:

$$g = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Reconstruct the original image.

Solution:

$$f = \tilde{H}^T g \tilde{H} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Localized error

Remark:

1. Haar Transform usually produces coefficient matrix with more zeros!

2. Localized error in coefficient matrix causes localized error in the reconstructed image

Elementary images under Haar transform:

Using Haar transform, f can be written as:

$$f = \tilde{H}^T \underbrace{g}_{\text{transformed image}} \tilde{H}$$

Let $\tilde{H} = \begin{pmatrix} - & \vec{h}_0^T & - \\ - & \vec{h}_1^T & - \\ & \vdots & \\ - & \vec{h}_{N-1}^T & - \end{pmatrix}$. Then:

$$f = \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} g_{ij} \underbrace{\vec{h}_i \vec{h}_j^T}_{= I_{ij}^H}$$

I_{ij}^T = elementary images under Haar Transform.

e.g. For 8×8 images, we have $64 = 8 \times 8$ elementary images:

$$\begin{array}{cccc} \vec{h}_0 \vec{h}_0^T & \vec{h}_0 \vec{h}_1^T & \cdots & \vec{h}_0 \vec{h}_7^T \\ \vec{h}_1 \vec{h}_0^T & \vec{h}_1 \vec{h}_1^T & \cdots & \vec{h}_1 \vec{h}_7^T \\ \vdots & \vdots & & \vdots \\ \vec{h}_7 \vec{h}_0^T & \vec{h}_7 \vec{h}_1^T & & \vec{h}_7 \vec{h}_7^T \end{array}$$

(each $\vec{h}_i \vec{h}_j^T \in M_{8\times 8}(\mathbb{R})$)

Recall:

- Image decomposition f $\quad f = \sum_{i=1}^{N} \sum_{j=1}^{N} g_{ij} \underbrace{I_{ij}}_{\text{elementary images}}$

① Storage saving

② Image processing by modifying transformed image (coefficient matrix)
(e.g. Removing coefficients associated to high-frequency elementary images)

- 2 Separable Image Transformation:

① SVD (elementary images not universal and meaningless)

② Haar (elementary images universal and meaningful) — unsmooth

Discrete Fourier Transform:

Definition:

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Proof of Inverse DFT:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\,\hat{g}(m,n) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)}\,\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M}+\frac{ln}{N}\right)}$$

$$= \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} g(k,l)\, e^{j2\pi\left(\frac{(p-k)m}{M}+\frac{(q-l)n}{N}\right)}$$

$$= \underbrace{\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{(p-k)m}{M}\right)}\sum_{n=0}^{N-1} e^{j2\pi\left(\frac{(q-l)n}{N}\right)}}_{(*)}$$

Note that:

$$\sum_{m=0}^{M-1} e^{j2\pi\left(\frac{mt}{M}\right)} = \underbrace{\frac{\left[e^{j2\pi\left(\frac{t}{M}\right)}\right]^{M}-1}{e^{j2\pi\left(\frac{t}{M}\right)}-1}}_{\text{if } t\neq 0} = M\,\delta(t) := \begin{cases} M & t=0 \\ 0 & t\neq 0 \end{cases}$$

$\therefore$ (*) becomes:

$$\frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} g(k,l)\, M\,\delta(p-k)\, N\,\delta(q-l) = g(p,q).$$

DFT in Matrix form

Theorem: Consider a $N \times N$ image g, the DFT of g can be written as:

$$\hat{g} = \mathcal{U} g \mathcal{U} \quad \text{(DFT in matrix form)}$$

where $\mathcal{U} = (\mathcal{U}_{k\ell})_{0 \le k, \ell \le N-1} \in M_{N \times N}$ and $\mathcal{U}_{k\ell} = \frac{1}{N} e^{-j \frac{2\pi k\ell}{N}}$.

Proof: Need to check $\hat{g}(k,\ell) = (\mathcal{U} g \mathcal{U})(k,\ell)$ (k-th row, ℓ-th col of $\mathcal{U} g \mathcal{U}$)

$$\text{LHS} = \hat{g}(k,\ell) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n)\, e^{-j2\pi\left(\frac{km}{N} + \frac{\ell n}{N}\right)}$$

$$\text{RHS}: \quad \mathcal{U} g \mathcal{U} = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) \begin{pmatrix} | \\ \vec{u}_m \\ | \end{pmatrix} \begin{pmatrix} - \vec{u}_n - \end{pmatrix}$$

$$\mathcal{U} = \begin{pmatrix} | & | & & | \\ \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_N \\ | & | & & | \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} - \vec{u}_1 - \\ - \vec{u}_2 - \\ \vdots \\ - \vec{u}_N - \end{pmatrix}$$

$$\vec{u}_m = \begin{pmatrix} u_{0m} \\ u_{1m} \\ \vdots \\ u_{N-1\,m} \end{pmatrix} = \frac{1}{N} \begin{pmatrix} e^{-j\frac{2\pi(0)m}{N}} \\ \vdots \\ e^{-j\frac{2\pi(k)m}{N}} \\ \vdots \\ e^{-j\frac{2\pi(N-1)m}{N}} \end{pmatrix}$$

$$\vec{u}_n = (u_{n0}, u_{n1}, \ldots, u_{nN}) = \frac{1}{N}\left(e^{-j\frac{2\pi(0)(n)}{N}}, \ldots, e^{-j\frac{2\pi(\ell)(n)}{N}}, \ldots\right)$$

$$\mathcal{U} g \mathcal{U}(k,\ell) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} g(m,n) \frac{e^{-j2\pi\frac{km}{N}}}{N} \cdot \frac{e^{-j2\pi\frac{\ell n}{N}}}{N}$$

$$= \text{LHS}$$

Theorem: $U^* U = \frac{1}{N} I$ where $U^* = (\overline{U})^T$ (conjugate transpose)

$U U^* = \frac{1}{N} I.$

$\therefore \quad U^{-1} = (NU)^*$

$(\overline{a+jb} = a-jb)$

$(\overline{e^{j\theta}} = \overline{\cos\theta + j\sin\theta} = \cos\theta - j\sin\theta = e^{-j\theta})$

Proof: Later !!

Image decomposition by DFT

Suppose $\hat{g} = \mathrm{DFT}(g) = \mathcal{U} g \mathcal{U}$

Then: $\mathcal{U}\mathcal{U}^* = \frac{1}{N} I = \mathcal{U}^*\mathcal{U}$

$\therefore \quad g = (N\mathcal{U})^* \hat{g} (N\mathcal{U})^*$

$\therefore \quad g = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{g}_{kl} \vec{w}_k \vec{w}_l^{\,T}$ Elementary image of DFT

where $\vec{w}_k = k^{th}$ col of $(N\mathcal{U})^*$