

## Lecture 2:

Recap:

### Definition: (Linear image transformation)

An image transformation  $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$  is linear if it satisfies:  
$$\mathcal{O}(af + bg) = a\mathcal{O}(f) + b\mathcal{O}(g) \text{ for all } f, g \in M_{N \times N}(\mathbb{R}), a, b \in \mathbb{R}.$$

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# Point Spread Function

Take  $f \in \mathcal{I} = M_{N \times N}(\mathbb{R})$ .

$$\text{Let } f = \begin{pmatrix} f(1,1) & \dots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(x,y) & \vdots \\ f(N,1) & \dots & f(N,N) \end{pmatrix} = \sum_{x=1}^N \sum_{y=1}^N \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & f(x,y) & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} = \sum_{x=1}^N \sum_{y=1}^N f(x,y) \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

Consider a linear image transformation  $\mathcal{U}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$ .

Let  $g = \mathcal{U}(f)$ . Then:

$$g(\alpha, \beta) = \left[ \sum_{x=1}^N \sum_{y=1}^N f(x,y) \mathcal{U} \left( \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \right) \right]_{\alpha, \beta}$$

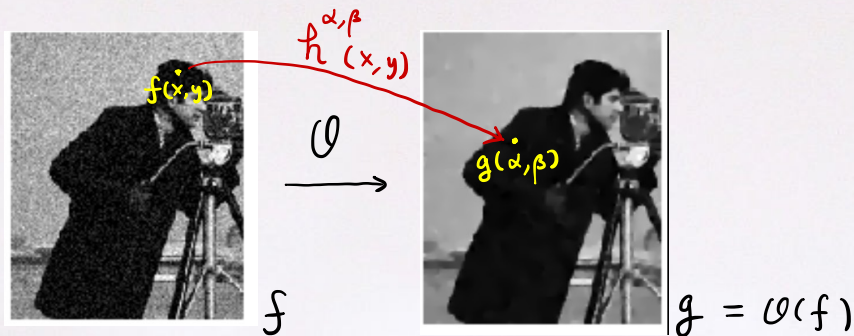
$$= \sum_{x=1}^N \sum_{y=1}^N f(x,y) h^{\alpha, \beta}(x,y)$$

where

$$h^{\alpha, \beta}(x,y) = [\mathcal{U}(P_{xy})]_{\alpha, \beta} ; P_{xy} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

$\downarrow y^{th}$   
 $\leftarrow x^{th}$

Remark:  $h^{\alpha, \beta}(x, y)$  determines how much the pixel value of  $f$  at  $(x, y)$  influences the pixel value of  $g$  at  $(\alpha, \beta)$ .



Definition: (Point spread function)

$h^{\alpha, \beta}(x, y)$  is usually called the point spread function (PSF)

## Separable linear image transformation

Definition: An image transformation  $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$  is said to be **separable** if there exists matrices  $A \in M_{N \times N}(\mathbb{R})$  and  $B \in M_{N \times N}(\mathbb{R})$  such that:  $\mathcal{O}(f) = A f B$  for all  $f \in M_{N \times N}(\mathbb{R})$ .

Theorem: Let  $\mathcal{O}$  be a separable image transformation given by:  $\mathcal{O}(f) = A f B$  for all  $f \in M_{N \times N}(\mathbb{R})$ , where  $A, B \in M_{N \times N}(\mathbb{R})$ . Then, the point spread function of  $\mathcal{O}$  is given by:

$$h^{\alpha, \beta}(x, y) = A(\alpha, x) B(y, \beta)$$

where  $A(\alpha, x)$  is the  $(\alpha, x)$  entry of  $A$ ,  $B(y, \beta)$  is the  $(y, \beta)$  entry of  $B$ .

## Periodic extension of images

Let  $f \in M_{N \times N}(\mathbb{R})$  be an image. We say  $f$  is periodically extended

if  $f(x, y) = f(x + pN, y + qN)$  where  $p, q$  are integers.

(So,  $f$  can now be defined on the entire plane  $-\infty < x, y < \infty$ )

For example,  $f(N+10, 2N+5) = f(10, 5)$

Geometrically, suppose  $f \in M_{3 \times 3}(\mathbb{R})$

|                                                                     |                                                                     |                                                                     |                                                                     |                                                                     |
|---------------------------------------------------------------------|---------------------------------------------------------------------|---------------------------------------------------------------------|---------------------------------------------------------------------|---------------------------------------------------------------------|
| $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ |
| $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ |
| $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ | $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ |

← row -2  
← row -1  
← row 0  
← row 1

$\therefore f(1, 5) = b$   
 $f(-2, 0) = c$



## Convolution

Definition: Consider  $k \in M_{N \times N}(\mathbb{R})$  and  $f \in M_{N \times N}(\mathbb{R})$ . Assume  $k$  and  $f$  are periodically extended. That is:

$$k(x, y) = k(x + pN, y + qN) \quad \text{where } p, q \text{ are integers.}$$
$$f(x, y) = f(x + pN, y + qN)$$

The convolution  $k * f$  of  $k$  and  $f$  is a  $N \times N$  matrix defined

as:

$$k * f(\alpha, \beta) = \sum_{x=1}^N \sum_{y=1}^N k(x, y) f(\alpha - x, \beta - y) \quad \text{for } 1 \leq \alpha, \beta \leq N$$

Example: Let  $k = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $f = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$ . Find  $k * f \in M_{2 \times 2}(\mathbb{R})$ .

The  $(1,1)$  entry of  $k * f$  is defined as:

$$\begin{aligned} k * f(1,1) &= \sum_{x=1}^2 \sum_{y=1}^2 k(x,y) f(1-x, 1-y) \\ &= \underbrace{k(1,1)}_{f(2,2)} \underbrace{f(0,0)}_{f(2,1)} + \underbrace{k(1,2)}_{f(2,1)} \underbrace{f(0,-1)}_{f(1,2)} + \underbrace{k(2,1)}_{f(1,2)} \underbrace{f(-1,0)}_{f(1,1)} + \underbrace{k(2,2)}_{f(1,1)} \underbrace{f(-1,-1)}_{f(1,1)} \\ &= (1)(1) + (2)(1) + (3)(2) + (4)(1) = 13. \end{aligned}$$

Similarly,  $k * f(1,2) = 14$ ,  $k * f(2,1) = 11$ ,  $k * f(2,2) = 12$ .

$$\therefore k * f = \begin{pmatrix} 13 & 14 \\ 11 & 12 \end{pmatrix}.$$



$f(3, 2)$

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$f(0, -1)$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} \leftarrow \text{Row } -2 \\ \leftarrow \text{Row } -1 \\ \leftarrow \text{Row } 0 \end{matrix}$$

$$f = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} \leftarrow \text{Row } 1 \\ \leftarrow \text{Row } 2 \\ \leftarrow \text{Row } 3 \end{matrix}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{matrix} \leftarrow \text{Row } 4 \\ \leftarrow \text{Row } 5 \\ \leftarrow \text{Row } 6 \end{matrix}$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

Col-2 Col-1 Col 0      Col 1 Col 2 Col 3      Col 4 Col 5 Col 6

Example: Let  $k = \begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix}$  and  $f \in M_{3 \times 3}(\mathbb{R})$ . Find  $k * f(2, 2)$

$$k * f(2, 2) = \sum_{x=1}^3 \sum_{y=1}^3 k(x, y) f(2-x, 2-y)$$

$$= \frac{1}{9} f(1, 1) + \frac{1}{9} f(1, 0) + \frac{1}{9} f(1, -1) + \frac{1}{9} f(0, 1) + \frac{1}{9} f(0, 0) + \frac{1}{9} f(0, -1) \\ + \frac{1}{9} f(-1, 1) + \frac{1}{9} f(-1, 0) + \frac{1}{9} f(-1, -1)$$

$$= \frac{f(1, 1) + f(1, 2) + f(1, 3) + f(2, 1) + f(2, 2) + f(2, 3) + f(3, 1) + f(3, 2) + f(3, 3)}{9}$$

(Averaging the intensity values in the neighborhood of  $f(2, 2)$ )

Remark: Averaging is commonly used in image processing, which is related to convolution.

Theorem: Let  $k \in M_{N \times N}(\mathbb{R})$ . Define  $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$  by:

$$\mathcal{O}(f) = k * f \quad \text{for all } f \in M_{N \times N}(\mathbb{R}).$$

Then:  $\mathcal{O}$  is linear.

Pf: Followed from the definition of convolution.

Theorem: Let  $k \in M_{N \times N}(\mathbb{R})$  and  $f \in M_{N \times N}(\mathbb{R})$ . Then:  $k * f = f * k$ .

Proof: Assuming  $\alpha, \beta > 1$ ,

$$k * f(\alpha, \beta) = \sum_{x=1}^N \sum_{y=1}^N k(x, y) f(\alpha - x, \beta - y)$$

$$= \sum_{\tilde{x}=\alpha-N}^{\alpha-1} \sum_{\tilde{y}=\beta-N}^{\beta-1} k(\alpha - \tilde{x}, \beta - \tilde{y}) f(\tilde{x}, \tilde{y}) \quad (\text{let } \tilde{x} = \alpha - x, \tilde{y} = \beta - y)$$

$$\stackrel{??}{=} \sum_{\tilde{x}=1}^N \sum_{\tilde{y}=1}^N k(\alpha - \tilde{x}, \beta - \tilde{y}) f(\tilde{x}, \tilde{y}) = f * k(\alpha, \beta).$$



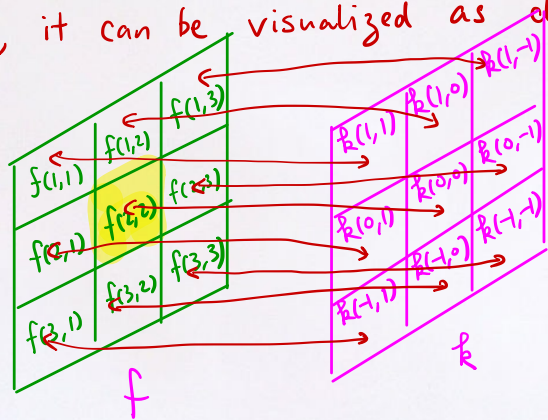
## Geometric meaning of convolution

Consider  $k \in M_{3 \times 3}(\mathbb{R})$  and  $f \in M_{3 \times 3}(\mathbb{R})$ .

$$\text{Consider: } k * f(2,2) = \sum_{x=1}^3 \sum_{y=1}^3 k(2-x, 2-y) f(x,y)$$

$$= k(1,1)f(1,1) + k(1,0)f(1,2) + k(1,-1)f(1,3) + k(0,1)f(2,1) + k(0,0)f(2,2) \\ + k(0,-1)f(2,3) + k(-1,1)f(3,1) + k(-1,0)f(3,2) + k(-1,-1)f(3,3)$$

Geometrically, it can be visualized as dot product:



Overlay  $k$  onto  $f$   
and take dot product.

Example: Let  $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$  be a linear image transformation defined by:  $\mathcal{O}(f)(\alpha, \beta) = f(\alpha+1, \beta) + 2f(\alpha, \beta) - 2f(\alpha-1, \beta) + f(\alpha, \beta+1) - 2f(\alpha, \beta-1)$  for all  $1 \leq \alpha, \beta \leq N$  and  $f \in M_{N \times N}(\mathbb{R})$ . Show that  $\mathcal{O}$  can be expressed in terms of a convolution.

Suppose  $\mathcal{O}(f) = k * f$  for some  $k \in M_{N \times N}(\mathbb{R})$ . Then,

$$\begin{aligned} \text{Then: } \mathcal{O}(f)(\alpha, \beta) &= \sum_{x=1}^N \sum_{y=1}^N k(\alpha-x, \beta-y) f(x, y) \\ &= \dots + \overset{1}{k(-1, 0)} f(\alpha+1, \beta) + \overset{2}{k(0, 0)} f(\alpha, \beta) + \overset{-2}{k(1, 0)} f(\alpha-1, \beta) + \overset{1}{k(0, -1)} f(\alpha, \beta+1) \\ &\quad + \overset{-2}{k(0, 1)} f(\alpha, \beta-1) + \dots \end{aligned}$$

We set  $k(-1, 0) = k(N-1, N) = 1$ ,  $k(0, 0) = k(N, N) = 2$ ,  $k(1, 0) = k(1, N) = -2$   
 $k(0, -1) = k(N, N-1) = 1$ ,  $k(0, 1) = k(N, 1) = -2$  and  $k(x, y) = 0$  otherwise.

Then:  $\mathcal{O}(f) = k * f$ .

Definition: The point spread function  $h^{\alpha, \beta}(x, y)$  of a linear image transformation is called **shift-invariant** if there exists a function  $\tilde{h}$

such that

$$h^{\alpha, \beta}(x, y) = \tilde{h}(\alpha - x, \beta - y)$$

for all  $1 \leq x, y, \alpha, \beta \leq N$ .

Remark: Given  $k \in M_{N \times N}(\mathbb{R})$ . Let  $\mathcal{U}$  be a linear image transformation defined by:  $\mathcal{U}(f) = k * f$  for all  $f \in M_{N \times N}(\mathbb{R})$ .

Then: the point spread function of  $\mathcal{U}$  is shift-invariant.

Let  $g = \mathcal{U}(f)$

$$g(\alpha, \beta) = \mathcal{U}(f)(\alpha, \beta) = \sum_{x=1}^N \sum_{y=1}^N \underset{h^{\alpha, \beta}(x, y)}{k(\alpha - x, \beta - y)} f(x, y)$$