

Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$$= \{ f \in M_{N \times N}(\mathbb{R}) : 0 \leq f(i, j) \leq M ; 1 \leq i, j \leq N \}$$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Remark: • Images are matrices (mathematically)

- For the ease of discussion, we assume $\mathcal{I} = M_{N \times N}(\mathbb{R})$
(collection of all $N \times N$ real matrices)

Image transformation = $\mathcal{O} : \mathcal{I} \rightarrow \mathcal{I}$ (transform one image to another)

Image processing v.s. Image Transformation

Let g be a "bad" image, which is a distorted version of a good (clean) image f . Then: we can write $g = \mathcal{O}(f)$, where $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$ transforms one image to another.

To solve the imaging problem:

(1) Design a suitable $T: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$ such that:

$$T(g) = f \quad (\text{i.e. } T \approx \mathcal{O}^{-1})$$

(2) Design mathematical method to solve:

$$g = \underbrace{\mathcal{O}(f)}_{\text{unknown}} \quad \left(\begin{array}{l} \text{Given } g \text{ and } \mathcal{O}, \text{ we solve for} \\ \text{the unknown } f \end{array} \right)$$

Definition: (Linear image transformation)

An image transformation $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$ is **linear** if it satisfies:
$$\mathcal{O}(af + bg) = a\mathcal{O}(f) + b\mathcal{O}(g) \text{ for all } f, g \in M_{N \times N}(\mathbb{R}), a, b \in \mathbb{R}.$$

Examples:

- Given $A \in M_{N \times N}(\mathbb{R})$. Define: $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$

by: $\mathcal{O}(f) = 2f + Af$ for all $f \in M_{N \times N}(\mathbb{R})$.

Then: $\mathcal{O}$ is linear.

- Given $A, B \in M_{N \times N}(\mathbb{R})$. Define $\mathcal{O}$ by:

$\mathcal{O}(f) = AfB$ for all $f \in M_{N \times N}(\mathbb{R})$. $\mathcal{O}$ is linear

- Given $A \in M_{N \times N}(\mathbb{R})$. Define $\mathcal{O}$ by:

$\mathcal{O}(f) = fAf$. Is $\mathcal{O}$ linear??

Point Spread Function

Take $f \in \mathcal{I} = M_{N \times N}(\mathbb{R})$.

$$\text{Let } f = \begin{pmatrix} f(1,1) & \dots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(x,y) & \vdots \\ f(N,1) & \dots & f(N,N) \end{pmatrix} = \sum_{x=1}^N \sum_{y=1}^N \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & f(x,y) & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} = \sum_{x=1}^N \sum_{y=1}^N f(x,y) \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

Consider a linear image transformation $\mathcal{U}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$.

Let $g = \mathcal{U}(f)$. Then:

$$g(\alpha, \beta) = \left[\sum_{x=1}^N \sum_{y=1}^N f(x,y) \mathcal{U} \left(\begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \right) \right]_{\alpha, \beta}$$

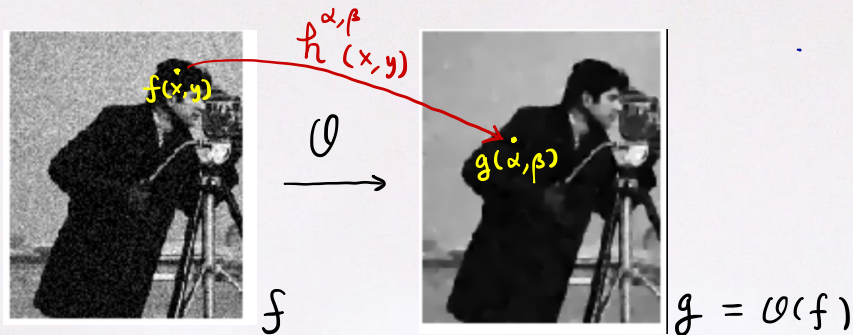
$$= \sum_{x=1}^N \sum_{y=1}^N f(x,y) h^{\alpha, \beta}(x,y)$$

where

$$h^{\alpha, \beta}(x,y) = [\mathcal{U}(P_{xy})]_{\alpha, \beta} ; P_{xy} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

$\downarrow y^{th}$
 $\leftarrow x^{th}$

Remark: $h^{\alpha, \beta}(x, y)$ determines how much the pixel value of f at (x, y) influences the pixel value of g at (α, β) .



Definition: (Point spread function)

$h^{\alpha, \beta}(x, y)$ is usually called the point spread function (PSF)

Example: Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Define: $\mathcal{U} : M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ by:

$$\mathcal{U}(f) = Af \quad \text{for all } f \in M_{2 \times 2}(\mathbb{R}).$$

Consider: $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then: $f = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$$\text{Then: } \mathcal{U}(f) = a \mathcal{U}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + b \mathcal{U}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) + c \mathcal{U}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) + d \mathcal{U}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= a \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{aligned} \therefore \mathcal{U}(f)(1,2) &= a \underbrace{\mathcal{U}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(1,1)} + b \underbrace{\mathcal{U}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(1,2)} + c \underbrace{\mathcal{U}\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(2,1)} + d \underbrace{\mathcal{U}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)(1,2)}_{h^{1,2}(2,2)} = 0 \end{aligned}$$

Separable linear image transformation

Definition: An image transformation $\mathcal{O}: M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$ is said to be **separable** if there exists matrices $A \in M_{N \times N}(\mathbb{R})$ and $B \in M_{N \times N}(\mathbb{R})$ such that: $\mathcal{O}(f) = A f B$ for all $f \in M_{N \times N}(\mathbb{R})$.

Example:

$$\begin{aligned} \bullet \quad \mathcal{O}(f) &= \alpha f \quad \text{for } \alpha \in \mathbb{R} \\ &= (\alpha I) f (I) \end{aligned}$$

- Discrete Fourier Transform is separable
- Discrete Haar Wavelet Transform is separable.

Remark: Separable image transformation is linear.

Theorem: Let $\mathcal{O}$ be a separable image transformation given by: $\mathcal{O}(f) = AfB$ for all $f \in M_{N \times N}(\mathbb{R})$, where $A, B \in M_{N \times N}(\mathbb{R})$. Then, the point spread function of $\mathcal{O}$ is given by:

$$h^{\alpha, \beta}(x, y) = A(\alpha, x) B(y, \beta)$$

where $A(\alpha, x)$ is the (α, x) entry of A , $B(y, \beta)$ is the (y, β) entry of B .

Proof: Let $g = \mathcal{O}(f) = AfB$. Then, the (α, β) entry of g is given by:

$$\begin{aligned} & \overset{\text{"}}{A(gB)(\alpha, \beta)} \\ & \left(\begin{matrix} \cancel{A(\alpha, 1)} & \cancel{A(\alpha, 2)} & \dots & \cancel{A(\alpha, N)} \end{matrix} \right) \left(\begin{matrix} (gB)(1, \beta) \\ (gB)(2, \beta) \\ \vdots \\ (gB)(N, \beta) \end{matrix} \right) \\ & = \sum_{x=1}^N A(\alpha, x) \sum_{y=1}^N f(x, y) B(y, \beta) \\ & = \sum_{x=1}^N \sum_{y=1}^N \underbrace{A(\alpha, x) B(y, \beta)}_{h^{\alpha, \beta}(x, y)} f(x, y) \end{aligned}$$