Lecture 10:

Recall:

- Mathematical formulation of image blur:

$$\underbrace{g}_{\text{Blurry image}} = \underbrace{f}_{\text{original image}} * h + \underbrace{n}_{\text{noise}} \Rightarrow \underbrace{G}_{\text{DFT of } g} = \overbrace{F}^{\text{DFT of } f} \odot \underbrace{H}_{= c\,DFT(h)} + \overbrace{N}^{\text{DFT of } n}$$

At $\forall (m,n)$,

$$G(m,n) = F(m,n)\,H(m,n)$$

$$\therefore F(m,n) = \frac{G(m,n)}{H(m,n)}$$

$\downarrow$ iDFT

$$\underline{\underline{f = iDFT(F)}}$$

- Direct inverse filtering: $T(u,v) = \dfrac{1}{H(u,v) + \varepsilon\, sgn(H(u,v))}$

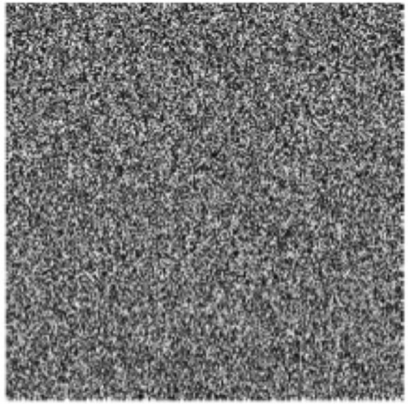
(Boast up noise)

- Modified inverse filtering:

$$B(u,v) = \frac{1}{1 + \left(\frac{u^2+v^2}{D^2}\right)^n} \quad \text{and} \quad T(u,v) = \frac{B(u,v)}{H(u,v) + \varepsilon\, sgn(H(u,v))}.$$

Original

Blurred image



Direct inverse filtering

Method 3: Wiener filter

Let $$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \frac{S_n(u,v)}{S_f(u,v)}}$$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \frac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\,G(u,v)$. Compute $\hat{f}(x,y)$ = inverse DFT of $\hat{F}(u,v)$.

$$T(u,v) = \frac{1}{H(u,v)} \underbrace{\left(\frac{|H(u,v)|^2}{|H(u,v)|^2 + K} \right)}_{\text{modifier}}$$

Wiener's filtering

High Noise | Medium Noise | Low Noise

<u>Method 4:</u> Constrained least square filtering

Disadvantages of Wiener's filter:

① $|N(u,v)|^2$ and $|F(u,v)|^2$ must be known / guessed

② Constant estimation of ratio is not always suitable

Goal: Consider a least square minimization model.

Let $g = h * f + n$

degradation   noise

In matrix form, $\vec{g} = D\vec{f} + \vec{n}$ $\vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}$, $D \in M_{N^2 \times N^2}$

$\vec{g} = \mathcal{S}(g)$, $\vec{f} = \mathcal{S}(f)$, $\vec{n} = \mathcal{S}(n)$

Stacked image of g

transformation matrix of $h * f$ (or f)

Constrained least square problem:

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} |\Delta f(x,y)|^2 \text{ subject to the constraint: } \|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

What is Δf ?

In the discrete case, we can estimate:

$$\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

Taylor expansion:

$$\frac{\partial^2 f}{\partial x^2}(x,y) \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \xrightarrow{\text{Put } h=1} \Delta f(x,y) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) \approx \frac{f(x,y+h) - 2f(x,y) + f(x,y-h)}{h^2}$$

$\therefore$ Δ is the Laplacian in the discrete case

Remark:

- More generally, $\Delta f = p * f$ ← discrete convolution

  where

$$p = \begin{array}{cc} \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ & & 1 & & \\ \vdots & 1 & -4 & 1 & \vdots \\ & & 1 & & \\ 0 & & \cdots & & 0 \end{pmatrix} & \begin{matrix} \\ \\ x=0 \\ \\ \\ \end{matrix} \\ \begin{matrix} \qquad\qquad y=0 \end{matrix} & \end{array}$$

Let L = transformation matrix representation the convolution with p.

Then: $\overrightarrow{\Delta f} = L\vec{f}$.

- Minimizing $\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} |\Delta f(x,y)|^2$ is to denoise. $\Rightarrow (L\vec{f})^T(L\vec{f}) = \underline{\vec{f}^T L^T L \vec{f}}$

- Also, $\vec{g} = D\vec{f} + \vec{n} \Leftrightarrow \vec{g} - D\vec{f} = \vec{n} \Rightarrow \|\vec{g} - D\vec{f}\|^2 = \|\vec{n}\|^2 \approx \varepsilon$

  noise level

$\therefore$ the constraint $\|\vec{g} - D\vec{f}\|^2$ is to solve the deblurring problem.

- $\displaystyle\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} \left|\Delta f(x,y)\right|^2$ $\leftarrow$ Denoise
- $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ $\leftarrow$ Deblur

<u>Remark:</u> $\|\vec{g} - D\vec{f}\|^2 = \varepsilon$ means we Allow some fixed level of noise.

$\shortparallel$
$\|\vec{n}\|^2$

Let $\overrightarrow{p*f} = S(p*f) = L\vec{f}$ transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$

We will prove:

<u>Theorem:</u> The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$\left(D^T D + \gamma L^T L\right)\vec{f} = D^T \vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\,\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma\,|P(u,v)|^2}\,G(u,v)$$

$\Big(H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & 1 & -4 & 1 & \vdots \\ & & 1 & & \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

$\Big)$

Remark: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{H(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

Sketch of proof:

Recall: our problem is to minimize:

$$\vec{f}^T L^T L \vec{f} \quad \text{subject to} \quad \|\vec{g} - D\vec{f}\|^2 = \varepsilon$$

$$\|\vec{g} - D\vec{f}\|^2 = (\vec{g} - D\vec{f})^T(g - D\vec{f})$$

From calculus, the minimizer must satisfy:

$$\nabla \mathcal{L} \overset{def}{=} \nabla\left(\vec{f}^T L^T L \vec{f} + \lambda(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f})\right) = 0 \quad \text{for}$$

where $\vec{f} = (f_1, f_2, \ldots, f_i, \ldots, f_{N^2})^T$ and λ is the Lagrange's multiplier.

Here, $\nabla \mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial f_1}, \frac{\partial \mathcal{L}}{\partial f_2}, \ldots, \frac{\partial \mathcal{L}}{\partial f_{N^2}}\right)^T$

Easy to check:

- $\nabla(\vec{f}^T \vec{a}) = \vec{a}$
- $\nabla(\vec{b}^T \vec{f}) = \vec{b}$
- $\nabla(\vec{f}^T A \vec{f}) = (A + A^T)\vec{f}$

$$\vec{f}^{\,T} \vec{a} = (f_1, f_2, \dots, f_n) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1 f_1 + a_2 f_2 + \dots + a_n f_n$$

$$\frac{\partial \vec{f}^{\,T} \vec{a}}{\partial f_j} = a_j$$

$$\therefore \nabla\left(\vec{f}^{\,T} \vec{a}\right) \overset{\text{def}}{=} \left(\frac{\partial \vec{f}^{\,T} \vec{a}}{\partial f_1}, \dots, \frac{\partial \vec{f}^{\,T} \vec{a}}{\partial f_n} \right)^T = (a_1, a_2, \dots, a_n)^T$$

etc . ..

$\therefore\ \mathcal{D} = 0 \Rightarrow (2L^T L)\vec{f} + \lambda(-D^T\vec{g} - D^T\vec{g} + 2D^T D\vec{f}) = 0$

$\Rightarrow (D^T D + \gamma L^T L)\vec{f} = D^T\vec{g}$ where $\gamma = \frac{1}{\lambda}$ and λ is the Lagrange's multiplier.

Parameter γ can be determined by direct substitution into the equation:

$$(\vec{g} - D\vec{f})^T(\vec{g} - D\vec{f}) = \varepsilon.$$

How about in the frequency (Fourier) domain?

Two important theorems

Notation: Let $\mathcal{O}$ be a linear transformation defined by:

$$\mathcal{O}(f) = k * f \quad \text{for all } f \in M_{N\times N}(\mathbb{R}), \text{ where}$$

$k \in M_{N\times N}(\mathbb{R})$.

Let $D \in M_{N^2 \times N^2}(\mathbb{R})$ be the transformation matrix representing $\mathcal{O}$.

That is, $\mathcal{S}(\mathcal{O}(f)) = D\, \mathcal{S}(f)$. $\in \mathbb{R}^{N^2}$

Here, $\mathcal{S}$ is the stacking operator. $\mathcal{S}(I)$ is the vectorized image of I

(1st col of I becomes first n entries of $\mathcal{S}(I)$, 2nd col of I becomes second n entries of $\mathcal{S}(I)$, ..., etc)

Theorem 1: Let $\mathbb{K} = DFT(k)$. Then:

$$D = W \Lambda_D W^{-1} \quad \text{and} \quad D^T = W \overline{\Lambda_D} W^{-1} \quad \text{where}$$

$$\Lambda_D = N^2 \begin{pmatrix} \mathbb{K}(0,0) & & & & & & & & \\ & \mathbb{K}(1,0) & & & & & & & \\ & & \ddots & & & & & & \\ & & & \mathbb{K}(N-1,0) & & & & & \\ & & & & \mathbb{K}(0,1) & & & & \\ & & & & & \ddots & & & \\ & & & & & & \mathbb{K}(N-1,1) & & \\ & & & & & & & \ddots & \\ & & & & & & & & \mathbb{K}(0,N-1) \\ & & & & & & & & & \ddots \\ & & & & & & & & & & \mathbb{K}(N-1,N-1) \end{pmatrix}$$

1st col of $\mathbb{K}$ becomes 1st n diagonal entries

2nd col of $\mathbb{K}$ becomes 2nd diagonal entries

for $W = W_N \otimes W_N$ where $W_N = \left(\frac{1}{\sqrt{N}} e^{\frac{2\pi j}{N} mn}\right)_{0 \le m,n \le N-1} \in M_{N \times N}(\mathbb{C})$

Example: Let $\mathcal{O}(f) = \underset{M_{2\times 2}}{k} * f$ where $K = DFT(k) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $D \in M_{4\times 4}(\mathbb{R})$ be the transformation matrix of $\mathcal{O}$.

Then: $D = W \Lambda_D W^{-1}$ where

$$\Lambda_D \underset{4}{=} \begin{pmatrix} a & & & 0 \\ & c & & \\ & & b & \\ 0 & & & d \end{pmatrix}$$

and $W = W_2 \otimes W_2$ $\left(W_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \right)$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} & \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} & -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix}$$

Theorem 2: Let $W = W_N \otimes W_N \in M_{N^2 \times N^2}$.

Let $f = (f_{mn})_{0 \le m,n \le N-1} \in M_{N \times N}(\mathbb{R})$. Consider $S(f) = \vec{f} \in \mathbb{R}^{N^2}$.

Then:

$$W^{-1}\vec{f} = N \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ F(0,1) \\ F(1,1) \\ \vdots \\ F(N-1,1) \\ \vdots \\ F(0,N-1) \\ F(1,N-1) \\ \vdots \\ F(N-1,N-1) \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} F \\ F \\ \vdots \\ F \end{matrix}}\right\} \text{1st col of } F \\ \left.\vphantom{\begin{matrix} F \\ F \\ \vdots \\ F \end{matrix}}\right\} \text{2nd col of } F \\ \vphantom{\vdots} \\ \left.\vphantom{\begin{matrix} F \\ F \\ \vdots \\ F \end{matrix}}\right\} N^{th} \text{ col of } F \end{array}$$

where $F = DFT(f)$

Example: Assume that:

$$g = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$W^{-1} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}$$

$$
W^{-1}\vec{g} = \frac{1}{3}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2}
\end{pmatrix}
\begin{pmatrix}
g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22}
\end{pmatrix}
$$

$$
= \frac{1}{3}
\begin{pmatrix}
g_{00} + g_{10} + g_{20} + g_{01} + g_{11} + g_{21} + g_{02} + g_{12} + g_{22} \\
g_{00} + g_{10}e^{-\frac{2\pi j}{3}} + g_{20}e^{-\frac{2\pi j}{3}2} + g_{01} + g_{11}e^{-\frac{2\pi j}{3}} + g_{21}e^{-\frac{2\pi j}{3}2} + g_{02} + g_{12}e^{-\frac{2\pi j}{3}} + g_{22}e^{-\frac{2\pi j}{3}2} \\
\vdots \\
\vdots \\
\vdots
\end{pmatrix}
\begin{matrix}
= 3^2 G(0,0) \\
= 3^2 G(1,0) \\
\vdots \\
\\
\\
\end{matrix}
$$

$G = \mathrm{DFT}(g)$

$\therefore \quad W^{-1}\vec{g} = 3\,\mathcal{S}(G)$

Suppose D is the transformation matrix representing the convolution with h.

(In other words, if $g = h * f$, then: $\underset{\in \mathbb{R}^{N^2}}{\vec{g}} = \underset{\in M_{N^2 \times N^2}}{D} \vec{f} \in \mathbb{R}^{N^2}$)

Let $H = DFT(h) \in M_{N \times N}$

Diagonalization of D:

$$D^T D = N^4 W \left[\underbrace{\begin{pmatrix} |H(0,0)|^2 & & & & & & & \\ & |H(1,0)|^2 & & & & & & \\ & & \ddots & & & & & \\ & & & |H(N-1,0)|^2 & & & & \\ & & & & |H(0,1)|^2 & & & \\ & & & & & \ddots & & \\ & & & & & & |H(N-1,1)|^2 & \\ & & & & & & & \ddots \\ & & & & & & & & |H(0,N-1)|^2 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & |H(N-1,N-1)|^2 \end{pmatrix}}_{\text{Stack } H \text{ to form the diagonal matrix.}} \right] W^{-1}$$

Suppose L is the transformation matrix representing the convolution with p

(In other words, if $g = p * f$, then: $\underset{\in \mathbb{R}^{N^2}}{\vec{g}} = \underset{\in M_{N^2 \times N^2}}{L} \vec{f}_{\in \mathbb{R}^{N^2}}$)

Let $P = DFT(p) \in M_{N \times N}$

Diagonalization of L:

$$L^T L = N^4 W \underbrace{\left[\begin{pmatrix} |P(0,0)|^2 & & & & & & & & \\ & |P(1,0)|^2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & |P(N-1,0)|^2 & & & & & \\ & & & & |P(0,1)|^2 & & & & \\ & & & & & \ddots & & & \\ & & & & & & |P(N-1,1)|^2 & & \\ & & & & & & & \ddots & \\ & & & & & & & & |P(0,N-1)|^2 \\ & & & & & & & & & \ddots \\ & & & & & & & & & & |P(N-1,N-1)|^2 \end{pmatrix} \right]}_{\text{Stack } P \text{ to form the diagonal matrix.}} W^{-1}$$

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} \,,$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1} \vec{f} = \cancel{W} \Lambda_D^* W^{-1} \vec{g}$$

$$\therefore \quad N^4 \begin{pmatrix} |H(0,0)|^2 + \gamma |P(0,0)|^2 & & & & \\ & |H(1,0)|^2 + \gamma |P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & |H(N-1,0)|^2 + \gamma |P(N-1,0)|^2 & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ \vdots \end{pmatrix} =$$

$$N^2 \begin{pmatrix} \overline{H(0,0)} & & & & \\ & \overline{H(1,0)} & & & \\ & & \ddots & & \\ & & & \overline{H(N-1,0)} & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} G(0,0) \\ G(1,0) \\ \vdots \\ G(N-1,0) \\ \vdots \end{pmatrix}$$

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2 + \gamma|P(u,v)|^2]NF(u,v) = N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2 \frac{|H(u,v)|^2 + \gamma|P(u,v)|^2}{\overline{H(u,v)}} F(u,v) = G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$

subject to the constraint that:

$$\| \underbrace{\vec{g} - H\vec{f}}_{\vec{n}} \|^2 = \varepsilon$$

(allow fixed amount of noise)

Image sharpening in the frequency domain

Goal: Enhance image so that it shows more obvious edges.

Method 1: Laplacian masking

Recall that : $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

In the discrete case, $\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x,y-1) + f(x-1,y) - 4f(x,y)$

or $\Delta f \approx p * f$ where $p = \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$

We can observe that $-\Delta f$ captures the edges of the image (leaving other region zero)

$\therefore$ Shapen image $= f + \overbrace{(-\Delta f)}^{\text{add more edges}}$

In the frequency domain: $\mathrm{DFT}(g) = \mathrm{DFT}(f) - \mathrm{DFT}(\overset{p*f}{\overset{\shortparallel}{\Delta f}})$

$= \mathrm{DFT}(f) - c\,\mathrm{DFT}(p) \odot \mathrm{DFT}(f)$

$\therefore \mathrm{DFT}(g)(u,v) = [1 - \underset{\underset{c\,\mathrm{DFT}(p)}{\shortparallel}}{H_{laplacian}(u,v)}]\, \mathrm{DFT}(f)(u,v)$