Real Analysis 24-11-22

• Vitali Convergence Thm.

Def. (Unif. Integrability)

Let $(f_n) \subset L^1(\mu)$. We say (f_n) is unif. integrable if

① $\sup_n \int |f_n| \, d\mu < \infty$.

② $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\int_E |f_n| \, d\mu < \varepsilon \text{ if } E \in \mathcal{M}, \ \mu(E) < \delta, \ n \in \mathbb{N}.$$

Thm 4.27 (Vitali Convergence Thm)

Let $\mu(X) < \infty$ and $f_n \in L^1(\mu)$, $n \geq 1$. Assume

① $f_n \to f$ a.e.

② (f_n) is uniformly integrable.

Then $f_n \to f$ in $L^1(\mu)$, i.e. $\int |f_n - f| \, d\mu \to 0$.

Proof. Let $\varepsilon > 0$. $\exists$ $\delta > 0$ such that

$$\sup_n \int_E |f_n| \, d\mu < \varepsilon \quad \text{if} \quad \mu(E) < \delta. \qquad (1)$$

By Fatou lemma, if $\mu(E) < \delta$, then

$$\int_E |f| \, d\mu = \int_E \lim_{n \to \infty} |f_n| \, d\mu$$

$$\leq \varliminf_{n \to \infty} \int_E |f_n| \, d\mu$$

$$< \varepsilon. \qquad (2)$$

Since $\mu(X) < \infty$, $f_n \to f$ a.e., by Egrov Thm,

$\exists$ $A \in \mathcal{M}$ with $\mu(A) < \delta$ such that

$$f_n \rightrightarrows f \quad \text{on} \quad X \setminus A.$$

Hence $\exists$ $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{if} \quad x \in X \setminus A, \; n \geq N.$$

Now for $n \geq N$,

$$\begin{aligned}
\int |f_n - f| \, d\mu &= \int_{X \setminus A} |f_n - f| \, d\mu + \int_A |f_n - f| \, d\mu \\
&\leq \varepsilon \cdot \mu(X \setminus A) + \int_A |f_n| \, d\mu \\
&\qquad + \int_A |f| \, d\mu \\
&\leq \varepsilon \cdot \mu(X) + 2\varepsilon \quad (\text{by (1) and (2)})
\end{aligned}$$

Hence we obtain the desired result.

$\blacksquare$

Chap 5. Radon - Nikodym Thm.

§ 5.1 Signed measures (符号测度)

Def. Let $(X, \mathcal{M})$ be a measurable space.

A function $\mu : \mathcal{M} \to \mathbb{R}$ is said to be a signed measure if

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) \quad \text{if } (E_n) \text{ is}$$

a partition of E, $\forall\ E \in \mathcal{M}$.

(i.e. $E_n \in \mathcal{M}$, $n \geq 1$ are disjoint subsets of E with $\bigcup_{n=1}^{\infty} E_n = E$)

Remark: (1) It is clear that $\mu(\emptyset) = 0$

also $|\mu(X)| < \infty$.

Hence a measure μ may be not a signed measure.

<u>Def</u>. Given a signed measure μ on $(X, \mathcal{M})$, the total variation of μ is defined by

$$|\mu|(E) := \sup\left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : (E_n) \text{ is a partition of } E \right\},$$

$\forall\, E \in \mathcal{M}$.

Remark: $\quad |\mu|(E_1) \leq |\mu|(E_2)$ if $E_1 \subset E_2$.

Prop 5.1 If μ is a signed measure on $(X, \mathcal{M})$, then its total variation $|\mu|$ is a finite measure on $(X, \mathcal{M})$.

Pf. First notice that $|\mu|(\emptyset) = 0$.

Next we prove that

$$|\mu|(E) = \sum_{n=1}^{\infty} |\mu|(E_n) \quad \text{if } (E_n) \text{ is a partition of } E.$$

Let us first prove the countable sub-additivity.

$$|\mu|(E) \le \sum_{n=1}^{\infty} |\mu|(E_n) \quad \text{if } (E_n) \text{ is a partition of } E.$$

Let (A_k) be a partition of E.

$$\begin{aligned}
\sum_{k=1}^{\infty} |\mu(A_k)| &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} \mu(A_k \cap E_n) \right| \\
&\le \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\mu(A_k \cap E_n)| \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\mu(A_k \cap E_n)| \\
&\le \sum_{n=1}^{\infty} |\mu|(E_n)
\end{aligned}$$

Taking supremum of all partitions (A_k) of E gives

$$|\mu|(E) \le \sum_{n=1}^{\infty} |\mu|(E_n).$$

Next we prove

$$|\mu|(E) \geq \sum_{n=1}^{\infty} |\mu|(E_n), \quad \forall \text{ a partition } (E_n) \text{ of } E$$

Clearly if $|\mu|(E_n) = \infty$ for some n, then

$|\mu|(E) \geq |\mu|(E_n) = \infty$, so the inequality holds

Now we assume that $|\mu|(E_n) < \infty$ for all n.

Then for each n, $\exists$ a partition $(E_n^k)_k$ of E_n such that

$$|\mu|(E_n) \leq \left(\sum_{k=1}^{\infty} |\mu(E_n^k)| \right) + 2^{-n} \cdot \varepsilon$$

Hence

$$\sum_{n=1}^{\infty} |\mu|(E_n) \leq \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\mu(E_n^k)| \right) + \varepsilon$$

(Notice $(E_n^k)_{1 \leq n \leq \infty,\ 1 \leq k < \infty}$ is a partition of E)

Hence $\sum_{n=1}^{\infty} |\mu|(E_n) \leq |\mu|(E) + \varepsilon$.

We obtain

$$\sum_{n=1}^{\infty} |\mu|(E_n) \leq |\mu|(E),$$

since $\varepsilon > 0$ is arbitrary.

Next we show that $|\mu|(X) < \infty$.

We need the following.

Lem 5.2. If $|\mu|(E) = \infty$ for some $E \in \mathcal{M}$, then $\exists$ $A, B \in \mathcal{M}$, $A \cup B = E$, A, B are djoint, such that

$$|\mu(A)|,\ |\mu(B)| \geq 1 \quad \text{and} \quad |\mu|(A) = \infty.$$

We postpone the proof of Lem 5.2 until we complete the proof of Prop 5.1.

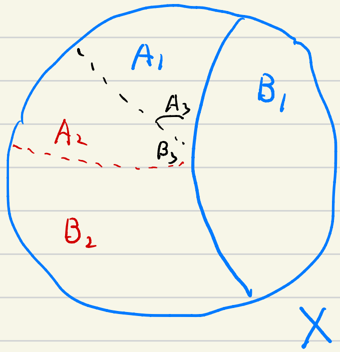
Now suppose on the contrary that

$$|\mu|(X) = \infty.$$

Then by Lem 5.2, we can find a partition $\{A_1, B_1\}$ of X

Such that

$$|\mu(A_1)| \geq 1, \quad |\mu(B_1)| \geq 1, \quad |\mu|(A_1) = \infty.$$



Using Lem 5.2 again, we can find a partition $\{A_2, B_2\}$ of A_1

Such that

$$|\mu(A_2)|, \ |\mu(B_2)| \geq 1, \quad |\mu|(A_2) = \infty.$$

Continuing this process, we can find a sequence of (B_n) such that they are disjoint, and

$$|\mu(B_n)| \geq 1, \quad \forall n.$$

Now take $B = \bigcup_{n=1}^{\infty} B_n$. Then

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n)$$

However, the series in the RHS diverges since

$$|\mu(B_n)| \not\to 0.$$

It leads to a contradiction. $\boxed{}$

Pf of Lem 5.2. Let $t > 0$.

Since $|\mu|(E) = \infty$, $\exists$ a partition (E_n) of E such that

$$\sum_{n=1}^{\infty} |\mu(E_n)| > t.$$

$\exists$ a large N such that

$$\sum_{n=1}^{N} |\mu(E_n)| > t.$$

We rearrange the sets E_n such that

$$\mu(E_1), \cdots, \mu(E_m) < 0$$

and

$$\mu(E_{m+1}), \cdots, \mu(E_N) \geq 0$$

Hence $$|\mu(E_1) + \cdots + \mu(E_m)| + |\mu(E_{m+1}) + \cdots + \mu(E_N)|$$
$$= \sum_{n=1}^{N} |\mu(E_n)| > t$$

WLOG, we assume that

$$|\mu(E_1) + \cdots + \mu(E_m)| > t/2.$$

Then take $A = E_1 \cup \cdots \cup E_m$

$B = E \setminus A.$

Clearly $|\mu(A)| > t/2,$

$$|\mu(B)| = |\mu(E) - \mu(A)| \geq |\mu(A)| - |\mu(E)|$$

$$\geq \frac{t}{2} - |\mu(E)|$$

Take a large t such that

$$\frac{t}{2} - |\mu(E)| > 1.$$

Then

$$|\mu(A)|,\ |\mu(B)| > 1.$$

Notice that

$$\infty = |\mu|(E) = |\mu|(A) + |\mu|(B)$$

So one of $|\mu|(A)$, $|\mu|(B)$ is ∞.

$\blacksquare$

Prop 5.3. Let $(X, \mathcal{M}, \mu)$ be a measure space.

Let $f \in L^1(\mu)$. Define

$$\lambda(E) = \int_E f \, d\mu, \quad E \in \mathcal{M}.$$

Then ① λ is a signed measure on $(X, \mathcal{M})$.

② The total variation $|\lambda|$ of λ satisfies

$$|\lambda|(E) = \int_E |f| \, d\mu, \quad E \in \mathcal{M}.$$

Pf. Clearly $\lambda : \mathcal{M} \to \mathbb{R}$ is well-defined.

Let $E \in \mathcal{M}$ and (E_n) be a partition of E.

Then $\chi_E = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \chi_{E_k}$

By the Dominated Convergence Thm,

$$\begin{aligned} \int_E f \, d\mu &= \int \chi_E f \, d\mu \\ &= \lim_{n\to\infty} \int \sum_{k=1}^{n} \chi_{E_k} f \, d\mu \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{E_k} f \, d\mu \end{aligned}$$

That is,

$$\lambda(E) = \lim_{n\to\infty} \sum_{k=1}^{n} \lambda(E_k)$$

$$= \sum_{k=1}^{\infty} \lambda(E_k).$$

Hence λ is a signed measure on $(X, \mathcal{M})$.

This proves (1).

Next we prove (2), i.e.

$$|\lambda|(E) = \int_E |f| \, d\mu, \quad E \in \mathcal{M}.$$

Let $E \in \mathcal{M}$. Let (E_n) be a partition of E

Then

$$\sum_{n=1}^{\infty} |\lambda(E_n)| = \sum_{n=1}^{\infty} \left| \int_{E_n} f \, d\mu \right|$$

$$\leq \sum_{n=1}^{\infty} \int_{E_n} |f| \, d\mu.$$

Taking supremum over all partitions (E_n) of E

gives

$$|\lambda|(E) \leq \sum_{n=1}^{\infty} \int_{E_n} |f| \, d\mu .$$

On the other hand, let

$$A = \{ x \in E : f(x) \geq 0 \},$$

$$B = \{ x \in E : f(x) < 0 \}.$$

Clearly $\{A, B\}$ is a partition of E.

$$|\lambda(A)| = \left| \int_A \underset{\frown}{f} \, d\mu \right|$$

$$= \int_A |f| \, d\mu$$

$$|\lambda(B)| = \left| \int_B \underline{f} \, d\mu \right|$$

$$= \left| \int_B (-f) \, d\mu \right|$$

$$= \int_B (-f) \, d\mu$$

$$= \int_B |f| \, d\mu$$

So

$$|\lambda(A)| + |\lambda(B)| = \int_A |f|\,d\mu + \int_B |f|\,d\mu$$

$$= \int (\chi_A + \chi_B)\,|f|\,d\mu$$

$$= \int \chi_E\,|f|\,d\mu$$

$$= \int_E |f|\,d\mu$$

Since

$$|\lambda|(E) \geq |\lambda(A)| + |\lambda(B)|$$

(since $\{A, B\}$ is a partition of E)

we obtain

$$|\lambda|(E) \geq \int_E |f|\,d\mu.$$

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