Real Analysis   24-11-01

§ 3.4 Hausdorff measures.

Def (Hausdorff, 1918)

Let $A \subset \mathbb{R}^n$, $\delta > 0$, $s \in [0, \infty)$, define

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} |A_i|^s : A \subset \bigcup_{i=1}^{\infty} A_i , \ |A_i| < \delta \right\}.$$

(here $|A_i| := \operatorname{diam}(A_i)$)

and

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A).$$

(Using the fact $\mathcal{H}^s_\delta(A) \uparrow \mathcal{H}^s(A)$ as $\delta \downarrow 0$)

We call $\mathcal{H}^s(A)$ the s-dimensional Hausdorff measure of A.

Thm 3.7. (a) $\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$ for each $s \geq 0$.

(b) Suppose $\mathcal{H}^s(A) < \infty$, then $\exists$ a Borel set B such that $B \supset A$ and

$$\mathcal{H}^s(B) = \mathcal{H}^s(A).$$

(c) For any open set $G \subset \mathbb{R}^n$,

$$\mathcal{H}^s(G) = \sup\{ \mathcal{H}^s(K) : K \text{ compact}, K \subset G \}.$$

(d) If A is a Borel set with $\mathcal{H}^s(A) < \infty$, then $\forall\, \varepsilon > 0$, $\exists$ compact $K \subset A$ such that

$$\mathcal{H}^s(A \setminus K) < \varepsilon.$$

Pf. (a). First we show that $\mathcal{H}^s_\delta$ is an outer measure.

This is clear since $\mathcal{H}^s_\delta$ is generated by a gauge $(\mathcal{R}_\delta, |\cdot|^s)$, where

$$\mathcal{R}_\delta = \{ A \subset \mathbb{R}^n : \operatorname{diam}(A) < \delta \}.$$

We claim that $\mathcal{H}^s$ is also an outer measure.

Clearly, $\mathcal{H}^s(\emptyset) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(\emptyset) = 0$.

Next for $A \subset \bigcup_{i=1}^{\infty} A_i$, then

$$\begin{aligned} \mathcal{H}^s_\delta(A) &\leq \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(A_i) \\ &\leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i). \end{aligned}$$

Letting $\delta \to 0$ gives $\mathcal{H}^s(A) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(A_i)$.

Hence $\mathcal{H}^s$ is an outer measure.

Next we show that $\mathcal{H}^s$ is a metric outer measure.

To see this, let $A, B \subset \mathbb{R}^n$ with $d(A,B) > 0$.

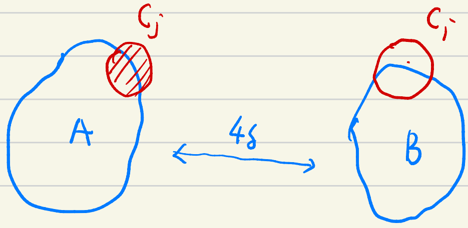
Take $0 < \delta < \dfrac{d(A,B)}{4}$.

Suppose $A \cup B \subset \bigcup_{k=1}^{\infty} C_k$ with $|C_k| < \delta$.

Write $\mathcal{A} = \{ C_j : C_j \cap A \neq \emptyset \}$

$\mathcal{B} = \{ C_j : C_j \cap B \neq \emptyset \}$.

Since $|C_j| < \delta$, $d(A,B) > 4\delta$, so no C_j intersects both A and B.



Hence $\sum_{k=1}^{\infty} |C_k|^s \geq \sum_{C_j \in \mathcal{A}} |C_j|^s + \sum_{C_j \in \mathcal{B}} |C_j|^s$

Notice that $\bigcup_{C_j \in \mathcal{A}} C_j \supset A$, $\bigcup_{C_j \in \mathcal{B}} C_j \supset B$.

So $\sum_{k=1}^{\infty} |C_k|^s \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$

Taking infimum over the covers $\{C_k\}$ of $A \cup B$ gives

$$\mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B),$$

so

$$\mathcal{H}^s_\delta(A \cup B) = \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B).$$

Letting $\delta \to 0$ gives

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Hence $\mathcal{H}^s$ is a metric outer measure. So it is a Borel measure.

This proves (a).

Now we prove (b): If $\mathcal{H}^s(A) < \infty$, then $\exists$ a Borel $B \supset A$ with $\mathcal{H}^s(B) = \mathcal{H}^s(A)$.

Notice that for $\delta > 0$, $\mathcal{H}^s_\delta(A) \leq \mathcal{H}^s(A) < \infty$.

Moreover for $C \subset \mathbb{R}^n$, $|C| = |\bar{C}|$, where $\bar{C}$ denotes the closure of C.

Hence for any integer $k > 0$, by definition, we can find $\{C^k_j\}_{j=1}^\infty$ such that C^k_j are closed sets

$|C_j^k| < \frac{1}{k}$, and $A \subset \bigcup_{j=1}^{\infty} C_j^k$, moreover

$$\sum_{j=1}^{\infty} |C_j^k|^s \leq \mathcal{H}^s_{1/k}(A) + \frac{1}{k} \leq \mathcal{H}^s(A) + \frac{1}{k}$$

Define $B_k = \bigcup_{j=1}^{\infty} C_j^k$, then B_k is Borel,

and $A \subset B_k$.

Let $B = \bigcap_{k=1}^{\infty} B_k$, then B is Borel, $B \supset A$.

Notice that for each $k \in \mathbb{N}$,

$$\begin{aligned} \mathcal{H}^s_{1/k}(B) &\leq \mathcal{H}^s_{1/k}(B_k) \\ &\leq \sum_{j=1}^{\infty} |C_j^k|^s \\ &\leq \mathcal{H}^s(A) + 1/k \end{aligned}$$

Letting $k \to \infty$ gives

$$\mathcal{H}^s(B) \leq \mathcal{H}^s(A),$$

and so $\mathcal{H}^s(B) = \mathcal{H}^s(A)$. This proves (b).

(c) For open $G \subset \mathbb{R}^n$,

(*) $\mathcal{H}^s(G) = \sup\{\mathcal{H}^s(K) : K \text{ compact}, K \subset G\}$.

To prove (*), it suffices to show that

$\exists$ a sequence of compact sets (K_j) such that

$$K_j \uparrow G$$

$\left(\text{i.e. } K_{j+1} \supset K_j \text{ and } G = \bigcup_{j=1}^{\infty} K_j \right)$

Then $\mathcal{H}^s(G) = \lim_{j\to\infty} \mathcal{H}^s(K_j)$ by the cty of measure

Now we construct such K_j as follows:

$$K_j = \{x \in \mathbb{R}^n : d(x, G^c) \geq \tfrac{1}{j},\ |x| \leq j\}.$$

A direct further check shows that $K_j \uparrow G$.

(d) If $\mathcal{H}^s(A) < \infty$, A is Borel, then $\forall\, \varepsilon > 0$, $\exists$ compact $K \subset A$ so that

$$\mathcal{H}^s(A \setminus K) < \varepsilon.$$

Actually this is a general property for all Borel measures on $\mathbb{R}^n$. You are referred to [Evans - Gariepy] Lem 1.1 (i), P. 6.

Prop 3.8. Let $A \subset \mathbb{R}^n$. Then

(1) $\mathcal{H}^s(TA) = \mathcal{H}^s(A)$ if T is a Euclidean motition (i.e. $Tx = Ux + b$, where U is an orthogonal transformation).

(2) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$, $\forall\, \lambda > 0$.

Prop 3.9. Let $A \subset \mathbb{R}^n$. Then

(1) $\mathcal{H}^s(A) = 0$, if $s > n$

(2) If $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$ if $t > s$.

(3) If $\mathcal{H}^s(A) > 0$, then $\mathcal{H}^t(A) = \infty$ if $t < s$.

Let $s > n$.

Pf. (1) We prove that $\mathcal{H}^s(\mathbb{R}^n) = 0$.

Notice that $\mathbb{R}^n$ is the countable union.

$$\bigcup_{z \in \mathbb{Z}^n} \left([0,1]^n + z \right)$$

It is enough to show that

$$\mathcal{H}^s\left([0,1]^n \right) = 0. \qquad (**)$$

Notice that for $k \in \mathbb{N}$, $[0,1]^n$ can be covered by k^n many subcubes of side $1/k$

Each such subcube is of diameter $\frac{\sqrt{n}}{k}$.

Hence

$$\mathcal{H}^s_{\sqrt{n}/k}([0,1]^n) \leq k^n \cdot \left(\frac{\sqrt{n}}{k}\right)^s$$

$$= (\sqrt{n})^s \cdot k^{n-s}$$

$$\to 0 \quad \text{as } k \to \infty$$

It follows that $\mathcal{H}^s([0,1]^n) = 0$.

(b) If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$ if $t > s$.

Assume $t > s$. Let $\delta > 0$. Then we can find a δ-cover $\{C_i\}_{i=1}^\infty$ of A such that

$$\sum_{i=1}^\infty |C_i|^s \leq \mathcal{H}^s_\delta(A) + 1.$$

Then $$\sum_{i=1}^\infty |C_i|^t = \sum_{i=1}^\infty |C_i|^s \cdot |C_i|^{t-s}$$

$$\leq \delta^{t-s} \cdot \sum_{i=1}^\infty |C_i|^s$$

$$\leq \left(\mathcal{H}^s_\delta(A)+1\right) \cdot \delta^{t-s}$$

So
$$\mathcal{H}^t_\delta(A) \le \sum_{i=1}^{\infty} |C_i|^t \le \left(\mathcal{H}^s_\delta(A)+1\right) \cdot \delta^{t-s}$$
$$\le \left(\mathcal{H}^s(A)+1\right) \delta^{t-s}.$$

Letting $\delta \to 0$ gives
$$\mathcal{H}^t(A) = 0.$$

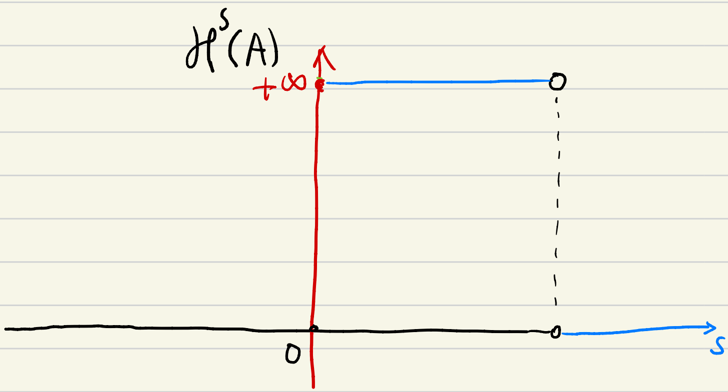


Figure. 1

Prop 3.10. (a) $\mathcal{H}^0$ is the counting measure on $\mathbb{R}^n$.

(b) $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$.

(c) $\mathcal{H}^n = C(n) \cdot \mathcal{L}^n$ on $\mathbb{R}^n$, where

$C(n)$ is a positive constant.

Pf. (a) follows from the definition.

(b) follows from the fact that

if $A \subset \mathbb{R}$, and $\{C_i\}$ is a cover of A,

then $\{[a_i, b_i]\}$ is also a cover of A

where $a_i = \inf C_i$, $b_i = \sup C_i$

and $\sum |C_i|^1 = \sum_i |b_i - a_i|$.

This property implies that $\mathcal{H}^1 = \mathcal{L}^1$, using the fact

$$\mathcal{L}^1(A) = \inf\left\{ \sum_i |b_i - a_i| : A \subset \bigcup_{i=1}^{\infty} [a_i, b_i],\ b_i - a_i < \delta \right\}$$

$\forall \delta > 0$.

(3) Since $\mathcal{H}^n$ is a translation invariant Borel measure, so $\exists$ $C(n)$ such that

$$\mathcal{H}^n = C(n)\,\mathcal{L}^n.$$

To see that $C(n)$ is a positive number, it is enough to show that

$$0 < \mathcal{H}^n([0,1]^n) < \infty.$$

By dividing $[0,1]^n$ into k^n many subcubes of side $\frac{1}{k}$ gives

$$\mathcal{H}^n_{\sqrt{n}/k}([0,1]^n) \leq k^n \cdot \left(\frac{\sqrt{n}}{k}\right)^n$$

$$\leq (\sqrt{n})^n < \infty$$

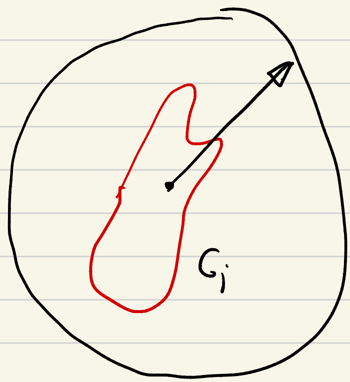
Letting $k \to \infty$ gives

$$\mathcal{H}^n([0,1]^n) \leq (\sqrt{n})^n.$$

Next we estimate ~~the~~ lower bound of $\mathcal{H}^n([0,1]^n)$.

Let $\{C_i\}$ be a δ-cover of $[0,1]^n$.

For each i, let B_i be a ball of radius $\operatorname{diam} C_i$ and so that $B_i \supset C_i$



Then

$$\sum_i |C_i|^n = 2^{-n} \sum_i |B_i|^n$$

$$= d_n 2^{-n} \cdot \sum_i \mathcal{L}^n(B_i)$$

$$\geq d_n 2^{-n} \cdot \mathcal{L}^n([0,1]^n)$$

$$= d_n 2^{-n}.$$

Hence

$$\mathcal{H}^n_\delta([0,1]^n) \geqslant d_n \cdot 2^{-n} > 0$$

$$\Rightarrow \mathcal{H}^n([0,1]^n) > 0.$$

$\square$

§ 3.5. Hausdorff dimension.

Def. Let $A \subset \mathbb{R}^n$. Define

$$\dim_H A := \sup\{ s \geq 0 : \mathcal{H}^s(A) > 0 \}$$
$$= \inf\{ s \geq 0 : \mathcal{H}^s(A) = 0 \}$$

We call it the Hausdorff dimension.

Facts: (1) If $A \subset B$, then $\dim_H A \leq \dim_H B$

(2) If $A = \bigcup_{i=1}^{\infty} A_i$ with A, A_i being Borel, then

$$\dim_H A = \sup_i \dim_H A_i$$

Def. A function $f: A \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be Hölder continuous with exponent α if $\exists$ $M > 0$ such that

$$(***) \qquad |f(x) - f(y)| \leq M \cdot |x-y|^{\alpha},$$

for all $x, y \in A$.

If $\alpha = 1$, then we call f is Lipschitz continuous.

Prop 3.11. Let $f: A \subset \mathbb{R}^n \to \mathbb{R}^n$ be Hölder cts with exponent α, and const M as in $(***)$.

Then for $s \geq 0$,

$$\mathcal{H}^{s/\alpha}(f(A)) \leq M^{s/\alpha} \cdot \mathcal{H}^s(A).$$

As a consequence, $\dim_H f(A) \leq \dim_H A / \alpha$.

Let $s \geq 0$.

Pf. Let $\delta > 0$. Let $\varepsilon > 0$.

Pick a δ-cover $\{C_j\}$ of A such that

$$\sum_j |C_j|^s \leq \mathcal{H}^s_\delta(A) + \varepsilon$$

Then $|f(C_j)| \leq M \cdot |C_j|^\alpha$ by the Hölder cty assumption on f.

Hence

$$\sum_j |f(C_j)|^{s/\alpha} \leq \sum_j M^{s/\alpha} \cdot \left(|C_j|^\alpha\right)^{s/\alpha}$$

$$= \sum_j M^{s/\alpha} \cdot |C_j|^s$$

$$\leq M^{s/\alpha} \cdot \left(\mathcal{H}^s_\delta(A) + \varepsilon\right)$$

But $\{f(C_j)\}_{j=1}^\infty$ is a $M \cdot \delta^\alpha$-cover of $f(A)$.

Hence

$$\mathcal{H}^{s/\alpha}_{M\cdot\delta^\alpha}(f(A)) \leq M^{s/\alpha}\cdot\left(\mathcal{H}^s_\delta(A)+\varepsilon\right).$$

Letting $\delta\to 0$, then letting $\varepsilon\to 0$, gives

$$\mathcal{H}^{s/\alpha}(f(A)) \leq M^{s/\alpha}\,\mathcal{H}^s(A).$$

$\blacksquare$

<u>Example 1</u>: Let $f:[0,1]\to\mathbb{R}$ be a Lipschitz function with Lip constant M. That is,

$$|f(x)-f(y)| \leq M|x-y|, \quad \forall\, x,y\in[0,1].$$

Let $G_f = \{(x, f(x)) : x\in[0,1]\} \subset \mathbb{R}^2$.

Then

$$1 \leq \mathcal{H}^1(G_f) \leq \sqrt{M^2+1}.$$

so $\dim_H G_f = 1$.

pf. Define $g : [0,1] \to G_f \subset \mathbb{R}^2$ by

$$x \mapsto (x, f(x)).$$

Then

$$|g(x) - g(y)| = \sqrt{(x-y)^2 + (f(x) - f(y))^2}$$

$$\leq \sqrt{M^2+1}\, |x-y|.$$

Applying Prop 3.11,

$$\mathcal{H}^{1/1}(g([0,1])) \leq \left(\sqrt{M^2+1}\right)^{1/1} \cdot \mathcal{H}^1([0,1])$$

$$\Rightarrow \mathcal{H}^1(G_f) \leq \sqrt{M^2+1}.$$

To see the other direction, notice that

$$g^{-1} : G_f \to [0,1], \quad (x, f(x)) \mapsto x.$$

Then $|g^{-1}(u) - g^{-1}(v)| \leq |u-v|$, $\forall\, u, v \in G_f$

(check it!)

Hence by Prop 3.11, (letting $s=1$, $d=1$)

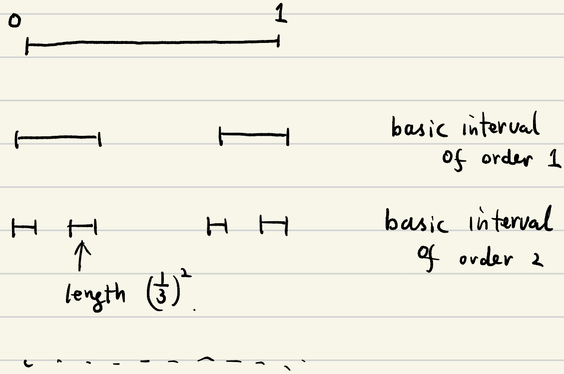
$$\mathcal{H}^{1/1}\left(g^{-1}(G_f)\right) \leq 1 \cdot \mathcal{H}^1(G_f)$$

That is, $1 = \mathcal{H}^1([0,1]) \leq \mathcal{H}^1(G_f)$.

$\boxed{/\!/}$

Example 2. Let C be the middle-third Cantor set. Find $\dim_H C$.

Solution :



From the construction of C, we see that for any $n \in \mathbb{N}$,

C can be covered by 2^n many basic intervals of order n

Each such interval has length 3^{-n}.

Hence
$$\mathcal{H}^s_{3^{-n}}(C) \leq 2^n \cdot (3^{-n})^s = 2^n \cdot 3^{-ns}$$

$$= 2^{n\left(1 - s\cdot(\log 3/\log 2)\right)}.$$

Letting $s = \log 2/\log 3$ gives

$$\mathcal{H}^{\log 2/\log 3}_{3^{-n}}(C) \leq 1.$$

$$\Rightarrow \mathcal{H}^{\log 2/\log 3}(C) \leq 1.$$

We claim that $\mathcal{H}^{\log 2/\log 3}(C) > 0.$

(Outline): Let μ be the Cantor measure,

i.e. μ is a prob. measure supported on C

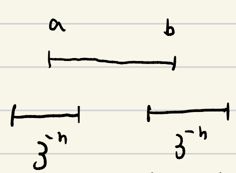
such that $\mu(I) = 2^{-n}$ for any basic interval

I of order n.

Notice that any interval $[a,b]$ with lengthen

between 3^{-n-1} and 3^{-n} intersects at most 2

basic intervals of order n.



Hence $\mu([a,b]) \leq 2\cdot 2^{-n} = 2\, 3^{-(\log 2/\log 3)n}$

$$= 2\cdot\left(3^{-n}\right)^{\log 2/\log 3}$$

$$\leq 2 \cdot \left(3(b-a)\right)^{\log 2/\log 3}$$

It follows that $\exists$ a constant $C > 0$ such that

$$\mu([a,b]) \leq d \cdot \underline{(b-a)}^{\log 2/\log 3}.$$

(****)

Let $\{A_i\}$ be a δ-cover of C.

Let $a_i = \inf A_i$, $b_i = \sup A_i$

Then $\{[a_i, b_i]\}$ is a δ-cover of C

with $\sum_i |A_i|^s = \sum_i |b_i - a_i|^s$

Let $s = \log 2/\log 3$. Then

$$\sum_i |A_i|^s = \sum_i |b_i - a_i|^s$$

$$\geq \frac{1}{d} \sum_i \mu([a_i, b_i]) \quad (\text{using } ****)$$

$$\geq \frac{1}{d} \mu(C) \geq \frac{1}{d}.$$

Hence $\mathcal{H}^s_\delta(C) \geq \frac{1}{d}$, $\forall \delta > 0$

Letting $\delta \to 0$ gives $\mathcal{H}^s(C) \geq \frac{1}{d} > 0$.