Real Analysis 24-09-27

Review

- Outer measure ($\mu(\emptyset)=0$, $\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$ if $A \subseteq \bigcup_{i=1}^{\infty} A_i$)

- (Caratheodory's Thm) Let μ be an outer measure on X.

Then (X, M_c, μ) is a **complete measure space**, where $M_c = \{ E \subset X : E \text{ is } \mu\text{-measurable} \}$.

§ 2.2 Topological and metric spaces.

- Topological spaces
- Metric spaces.

Prop. 2.3 Let (X, M, μ) be a measure space, where X is a topological space and assume that

$$M \supseteq \mathcal{B}_X \quad (\text{where } \mathcal{B}_X \text{ is the Borel } \sigma\text{-algebra on } X)$$

Then any continuous function $f: X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) is $\mathcal{M}$-measurable.

Pf. $\forall$ open $G \subset \mathbb{R}$ (or $\overline{\mathbb{R}}$), by continuity,

$f^{-1}(G)$ is open in X

hence $f^{-1}(G) \in \mathcal{B}_X \subset \mathcal{M}$. $\boxed{}$.

Def. (Borel measure)

An outer measure μ on a topological space X is said to be a Borel measure if all Borel sets are μ-measurable.

Prop 2.4 (Caratheodory's criterion)

Let (X, d) be a metric space. Let μ be an outer measure on X. Suppose that μ satisfies

(*) $\mu(A \cup B) = \mu(A) + \mu(B)$ if $d(A, B) > 0$,

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

(An outer measure μ satisfying $(*)$ is called a metric outer measure)

Then μ is a Borel measure.

Proof. It suffices to show that all closed sets in X are μ-measurable.

Let A be a closed set. We need to show

$$\mu(C) \geq \mu(C \cap A) + \mu(C \setminus A), \quad \forall\, C \subset X.$$

For $n \in \mathbb{N}$, define

$$A_n = \left\{ x \in X : \ d(x, A) \leq \frac{1}{n} \right\},$$

where $d(x, A) := \inf \{ d(x,y) : y \in A \}$.

Then A_n are closed, $A_n \searrow A$.

Notice that $d(A, A_n^c) \geq \frac{1}{n}$. So

$$d(C \cap A,\ C \setminus A_n) \geq \frac{1}{n}.$$

Hence $\mu(C) \geq \mu\big((C \cap A) \cup (C \setminus A_n) \big)$

$$= \mu(C \cap A) + \mu(C \setminus A_n), \quad \forall\, n.$$

Next we show that

$$(**) \qquad \varliminf_{n\to\infty} \mu(C\setminus A_n) \geq \mu(C\setminus A),$$

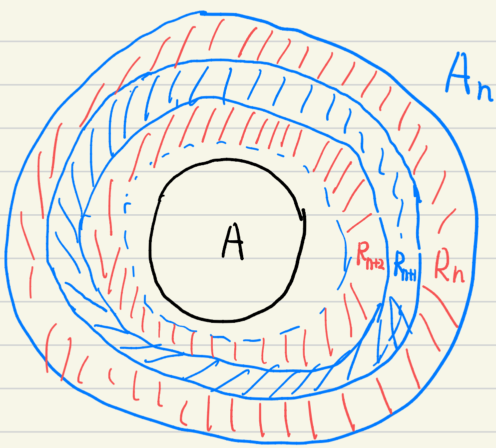
which implies $\mu(C) \geq \mu(C\cap A) + \mu(C\setminus A)$.

To prove $(**)$, define for $k\in\mathbb{N}$,

$$R_k = \left\{ x\in X : \frac{1}{1+k} < d(x,A) \leq \frac{1}{k} \right\}.$$

Then

$$A_n = A \cup \left(\bigcup_{k=n}^{\infty} R_n \right) \quad \text{with union being disjoint.}$$



Then $A^c = A_n^c \cup (A_n\setminus A)$

So

$$C \setminus A \subset (C \setminus A_n) \cup (C \cap (A_n \setminus A))$$

$$= (C \setminus A_n) \cup \left(C \cap \left(\bigcup_{k=n}^{\infty} R_k\right)\right).$$

Hence

$$\mu(C \setminus A) \leq \mu(C \setminus A_n) + \mu\left(C \cap \left(\bigcup_{k=n}^{\infty} R_k\right)\right).$$

To show that $\varliminf_{n \to \infty} \mu(C \setminus A_n) \geq \mu(C \setminus A)$,

it is enough to show

$$\sum_{k=1}^{\infty} \mu(C \cap R_k) < \infty$$

(which implies $\mu(C \cap \bigcup_{k=n}^{\infty} R_k) \to 0$ as $n \to \infty$.)

Notice that $R_2, R_4, \cdots, R_{2k}, \cdots,$ have positive distance between them, So are $R_1, R_3, R_5, \ldots$

Hence

$$\begin{aligned}
&\mu(C) \\
&\geq \mu\big((C \cap R_2) \cup (C \cap R_4) \cup \cdots (C \cap R_{2k})\big) \\
&= \mu(C \cap R_2) + \mu\big((C \cap R_4) \cup \cdots \cup (C \cap R_{2k})\big) \\
&= \cdots \\
&= \mu(C \cap R_2) + \cdots + \mu(C \cap R_{2k}).
\end{aligned}$$

Hence

$$\sum_{k=1}^{\infty} \mu(C \cap R_{2k}) \leq \mu(C) < \infty.$$

Similarly

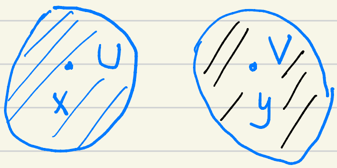
$$\sum_{k=1}^{\infty} \mu(C \cap R_{2k-1}) \leq \mu(C) < \infty$$

Therefore $\displaystyle\sum_{k=1}^{\infty} \mu(C \cap R_k) \leq 2 \cdot \mu(C) < \infty$ $\boxed{}$

§ 2.3 Locally compact Hausdorff spaces.

Def. A topological space X is said to be a Hausdorff space if $\forall$ $x, y \in X$ with $x \neq y$, $\exists$ open sets U and V such that

$$x \in U, \quad y \in V \quad \text{and} \quad U \cap V = \emptyset.$$



Def. A topological space is said to be locally compact if $\forall$ $x \in X$, $\exists$ an open U such that

$x \in U$ and $\overline{U}$ is compact.

(closure of U).

Notation: X is said to be a LCHS if X is a Hausdorff space and locally compact.

Remark:

- $\mathbb{R}^n$ is a LCHS.
- All compact metric spaces are LCHS.

Prop 2.5. Let X be a LCHS. Let $K \subset G$ where K is compact, G is open in X.

Then $\exists$ open V such that $\overline{V}$ is compact and

$$K \subset V \subset \overline{V} \subset G.$$

Thm 2.6 (Urysohn's lemma).

Let X be a LCHS.

Let $K \subset G$, K compact, G open.

Then there exists a cts $f : X \to \mathbb{R}$ such that

- $\text{supp}(f)$ is a compact subset of G.
- $0 \leq f \leq 1$ on X
- $f(x) = 1$ for all $x \in K$

where

$$\text{supp}(f) = \overline{\{x : f(x) \neq 0\}}.$$

Thm 2.7 (partition of Unity).

Let X be a LCHS.

Suppose $K \subset \bigcup_{k=1}^{N} G_k$, with G_k open

where K is compact.

There exist $\{\varphi_j\}_{j=1}^{N} \subset C(X)$,

such that

$$\varphi_j < G_j \quad \text{and} \quad \sum_{j=1}^{N} \varphi_j = 1 \text{ on } K$$

where $\varphi_j < G_j$ means that

① $\operatorname{supp}(\varphi_j)$ is a compact subset of G_j

② $0 \leq \varphi_j \leq 1$ on X.

§ 2.4 Riesz representation Thm.

- For a topologial space X, let

$$C_c(X) = \{ f \in C(X) : \operatorname{supp}(f) \text{ is compact} \}.$$

clearly, $C_c(X)$ is a vector space.

(that is, $af + bg \in C_c(X)$ if $f, g \in C_c(X)$ and $a, b \in \mathbb{R}$)

- A linear functional on a vector space is simply a linear map from the vector space to $\mathbb{R}$.

- A linear functional Λ on $C_c(X)$ is called positive if

$$\Lambda(f) \geq 0 \quad \text{if} \quad f \geq 0.$$

Example: Let X be a topological space. Let μ be a Borel measure on X such that $\mu(K) < \infty$ for all compact sets K.

Define

$$\Lambda(f) = \int_X f \, d\mu, \quad \forall f \in C_c(X).$$

Then Λ is a positive linear functional on $C_c(X)$.

Justification: It is easy to show the positivity and linearity of Λ.

Below we show that $\Lambda(f) \in \mathbb{R}$ for $f \in C_c(X)$.

Let $f \in C_c(X)$ and $K = \operatorname{Supp}(f)$.

Then K is compact.

Hence

$$\sup_{x \in X} |f(x)| = \sup_{x \in K} |f(x)| < \infty.$$

It follows that

$$\int_X f \, d\mu = \int_K f \, d\mu.$$

and $$\left| \int_K f \, d\mu \right| \leq \int_K |f| \, d\mu$$

$$\leq \mu(K) \cdot \sup_{x \in K} |f(x)|$$

$$< \infty.$$

Thm 2.8 (Riesz representation Thm)

Let X be a LCHS. Let Λ be a positive linear functional on $C_c(X)$.
Then $\exists$ a Borel measure μ on X such that μ is finite on every compact set, and

$$\Lambda(f) = \int f \, d\mu, \qquad \forall\ f \in C_c(X).$$

- Before the proof, we construct a measure μ from Λ.

Let $G \subset X$ be non-empty and open. We define

$$\mu_0(G) = \sup\{ \Lambda(f) : f < G \}.$$

(Recall $f < G$ means that $f \in C_c(X)$, $0 \leq f \leq 1$, $\operatorname{supp}(f) \subset G$.
By Urysohn Lem, $\exists\ f \in C_c(X)$ such that $f < G$)

Next set $\mu_0(\emptyset) = 0$.

Now for any $E \subset X$, define

$$\mu(E) = \inf\{\mu_0(G) : G \text{ is open, } G \supset E\}.$$

Proof of Riesz representation Thm (Thm 2.8):

First observe that

$$\mu_0(G_1) \leq \mu_0(G_2) \text{ for open set } G_1, G_2 \text{ with } G_1 \subset G_2.$$

As a direct consequence, we have

(i) $\mu(G) = \mu_0(G)$ for open $G \subset X$.

(ii) $\mu(E_1) \leq \mu(E_2)$ if $E_1 \subset E_2$.

Next we prove the theorem in 4 steps.

① μ is an outer measure.

② all Borel sets are μ-measurable.

③ $\mu(K) < \infty$ for compact K.

④ $\Lambda(f) = \int f \, d\mu, \quad f \in C_c(X)$.

Step 1. μ is an outer measure.

We need to show that

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j) \text{ if } E \subset \bigcup_{j=1}^{\infty} E_j.$$

We may assume $\sum_{j=1}^{\infty} \mu(E_j) < \infty$.

Let $\varepsilon > 0$. Pick open $G_j \supset E_j$ such that

$$\mu(E_j) > \mu_0(G_j) - \frac{\varepsilon}{2^j}, \quad j = 1, 2, \ldots$$

Set $G = \bigcup_{j=1}^{\infty} G_j$. Then G is open.

Now we estimate $\mu_0(G)$. Let $f \prec G$.

Let $K = \text{supp}(f)$. Then K is compact.

Since $K \subset G = \bigcup_{j=1}^{\infty} G_j$, by the compactness of K,

$\exists N$ such that

$$K \subset \bigcup_{j=1}^{N} G_j .$$

Then by the theorem of partition of unity,

$\exists \quad \varphi_j \prec G_j$ such that

$$\sum_{j=1}^{N} \varphi_j = 1 \quad \text{on } K.$$

We obtain that

$$f = \sum_{j=1}^{N} f \cdot \varphi_j \quad \text{on } X.$$

Hence

$$\begin{aligned} \Lambda(f) &= \sum_{j=1}^{N} \Lambda(f\varphi_j) \\ &\leq \sum_{j=1}^{N} \mu_0(G_j) \quad (\text{since } f\varphi_j \prec G_j) \\ &\leq \sum_{j=1}^{\infty} \mu_0(G_j). \end{aligned}$$

Since f is arbitrarily taken with $f \prec G$, we obtain

$$\mu_0(G) \leq \sum_{j=1}^{\infty} \mu_0(G_j)$$

Hence

$$\begin{aligned}\mu(E) \leq \mu_0(G) &\leq \sum_{j=1}^{\infty} \mu_0(G_j) \\ &\leq \sum_{j=1}^{\infty} \left(\mu(E_j) + \frac{\varepsilon}{2^j} \right) \\ &= \left(\sum_{j=1}^{\infty} \mu(E_j) \right) + \varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$ gives

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

Step 2. μ is a Borel measure.

Equivalently, we need to show that all open sets are μ-measurable.

Let $U \subset X$ be open. We need to prove

$$\mu(C) \geq \mu(C \cap U) + \mu(C \setminus U), \quad \forall C \subset X.$$

By the definition of μ, it is enough to prove

$$(*) \quad \mu(G) \geq \mu(G \cap U) + \mu(G \setminus U), \quad \forall \text{ open } G$$

(because if this is true, then $\forall\, \varepsilon > 0$, pick open $G \supset C$ such that $\mu(C) \geq \mu(G) - \varepsilon$. Then by $(*)$,

$$\begin{aligned} \mu(C) \geq \mu(G) - \varepsilon &\geq \mu(G \cap U) + \mu(G \setminus U) - \varepsilon \\ &\geq \mu(C \cap U) + \mu(C \setminus U) - \varepsilon. \end{aligned}$$)

To prove $(*)$, we may assume $\mu(G) < \infty$.
Let $\varepsilon > 0$, and pick $\varphi < G \cap U$ such that

$$\Lambda(\varphi) \geq \mu_0(G \cap U) - \varepsilon.$$

Let $K = \operatorname{supp}(\varphi)$. Pick $\psi < G \setminus K$.

Since $\operatorname{supp}(\psi)$ and K are disjoint,

$$\varphi + \psi < G.$$

Hence

$$\begin{aligned} \mu(G) = \mu_0(G) &\geq \Lambda(\varphi + \psi) \\ &= \Lambda(\varphi) + \Lambda(\psi) \\ &\geq \mu(G \cap U) - \varepsilon + \Lambda(\psi). \end{aligned}$$

Recall that $\psi < G \setminus K$ is arbitrily taken, we obtain

$$\mu(G) \geq \mu(G \cap U) - \varepsilon + \mu(G \setminus K)$$

$$\geq \mu(G \cap U) - \varepsilon + \mu(G \setminus U)$$

(since $G \setminus K \supset G \setminus U$).

Letting $\varepsilon \to 0$ gives

$$\mu(G) \geq \mu(G \cap U) + \mu(G \setminus U).$$

Step 3. μ is finite on compact sets.

We shall prove

$$\mu(K) = \inf\{\Lambda(f) : K \prec f\}$$

for all compact sets K,

where $K \prec f$ means $f \in C_c(X)$,

$0 \leq f \leq 1$ on X and $f = 1$ on K.

We first show $\mu(K) \leq \inf\{\Lambda(f) : K \prec f\}$.

Let $f \in C_c(X)$ such that $K \prec f$.

For $\alpha \in (0,1)$, define

$$G_\alpha = \{x \in X : f(x) > \alpha\}.$$

Then G_α is open and $G_\alpha \supset K$.

Let $\varphi \prec G_\alpha$. Then

$$\varphi < \frac{f}{\alpha} \quad \text{on } G_\alpha$$

Hence

$$\varphi \leq \frac{f}{\alpha} \quad \text{on } X.$$

It follows that

$$\Lambda(\varphi) \leq \Lambda\left(\frac{f}{\alpha}\right) = \frac{1}{\alpha}\Lambda(f)$$

(here we used the positivity of Λ)

Hence

$$\mu(G_\alpha) \leq \frac{1}{\alpha}\Lambda(f).$$

In particular

$$\mu(K) \leq \mu(G_\alpha) \leq \frac{1}{\alpha} \Lambda(f).$$

Letting $\alpha \uparrow 1$ gives $\mu(K) \leq \Lambda(f)$.

This showes $\mu(K) \leq \inf\{\Lambda(f) : K < f\}$.

To show the other direction, $\forall \varepsilon > 0$,

we can find open $G \supset K$ such that

$$\mu(K) \geq \mu_0(G) - \varepsilon = \mu(G) - \varepsilon.$$

By Urysohn's lemma, $\exists$ $f \in C_c(X)$ such that

$$K < f < G.$$

Hence

$$\begin{aligned}
\mu(K) &\geq \mu(G) - \varepsilon \\
&\geq \Lambda(f) - \varepsilon. \\
&\geq \inf\{\Lambda(g) : K < g\} - \varepsilon
\end{aligned}$$

Letting $\varepsilon \to 0$ gives the desired inequality.

Step 4. $\Lambda(f) = \int f \, d\mu$, $\forall f \in C_c(X)$.

(To be proved in the next class).