Network Calculus

- Overview
- Mathematical Model (with applications)
- Some of the proof

Part I - Overview

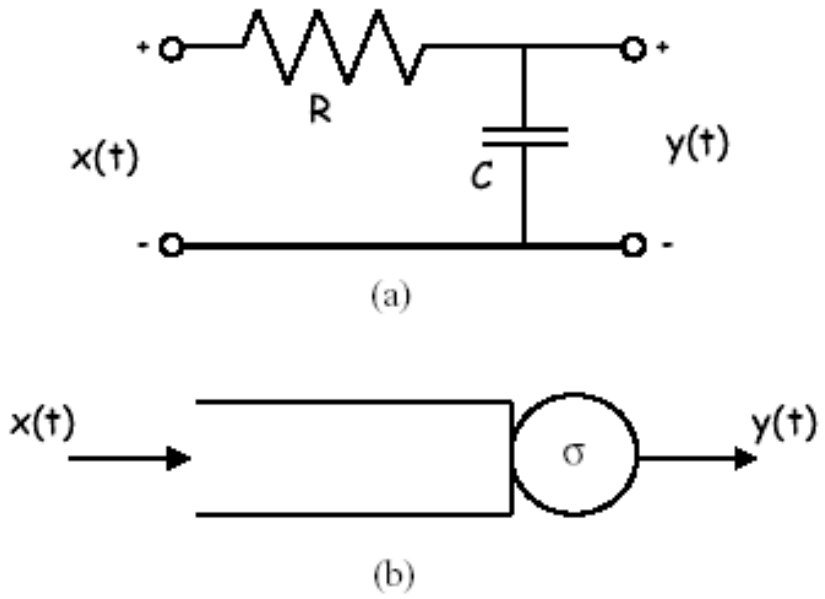
- What is Network Calculus
- Motivations
- References

What is Network Calculus

- Mathematical modeling of networks
- Analogous to system theory
- Using min-plus algebra
    - Deal with functions with special operators
  - Deterministic approach

Analogous

- Graph of analogous to system theory



Motivations

- Get a better understanding of the network behavior
- Isolate different network components, analyze each one independently
- Model some scenario and make good estimation
  - buffer bound
  - delay bound

References

- Text Book: *Network Calculus – A Theory of Deterministic Queuing Systems*
  - http://icawww1.epfl.ch/PS_files/NetCal.htm
- References
  - http://www.cis.ohio-state.edu/~durresi/presentations/calc_ad.pdf
  - http://icawww1.epfl.ch/PS_files/netCalIntroSlidesA.pdf

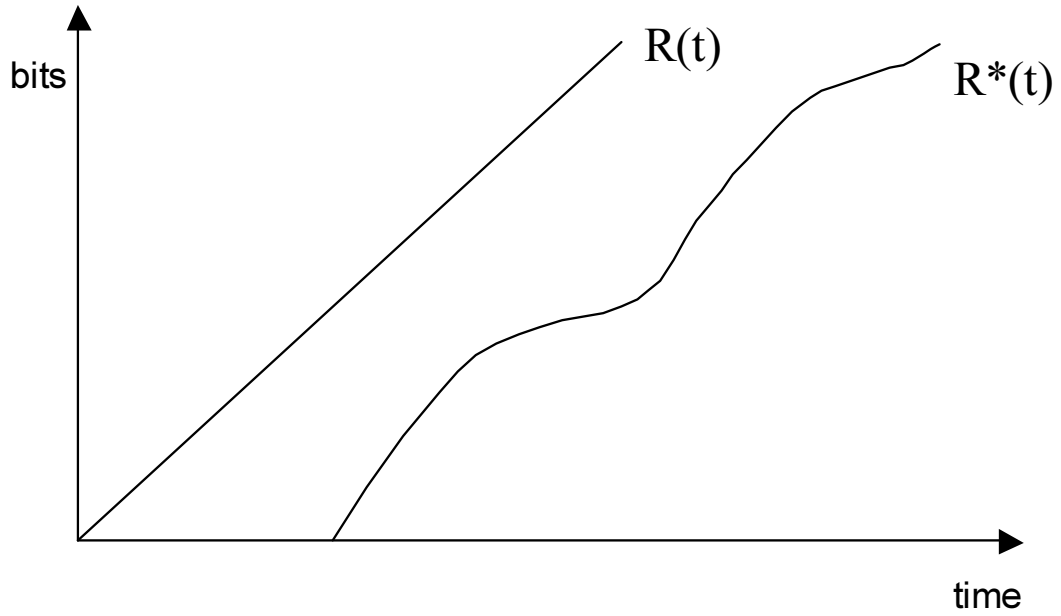
Part II - Mathematical Modeling

- Arrival function R and Departure function R^*
- Arrival curve
- Service curve
- Important Bounds
- Concatenation of Network Elements
- Other applications (Greedy Shaper ..)

Arrival R and departure R^*

- Definition
  - *R(t) is the arrival function, namely the number of bits seen on the input flow in time interval [0,t].*
  - *R^*(t) is the departure function, namely the cumulative function at the output of the system*
- Why use cumulative function?
  - Information of the traffic
  - Special feature

Arrival R and departure R^*



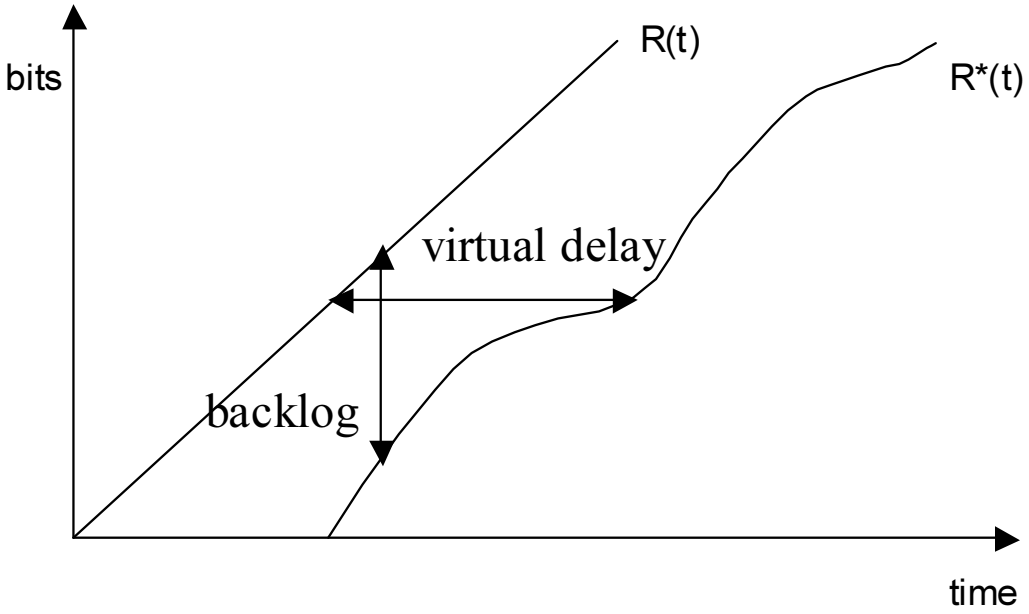
Arrival R and departure R^*

- Backlog
  - *The backlog in the system at time t, $x(t)=R(t)-R^*(t)$*
  - Property : Vertical Deviation between *R(t)* and $R^*(t)$
  - Relationship with the buffer of a system.
- Virtual delay
  - *The virtual delay at time t,*

    $d(t)=inf\{s\geq 0: R(t)\leq R^*(t+s)\}$
  - Property : Horizontal Deviation between R(t) and R^*(t)

Backlog and Virtual Delay

- The graph of backlog and virtual delay.



Arrival Curve

- Motivation
- Definition
- Samples
- Min-Plus Convolution
- Good Function
- Matlab results

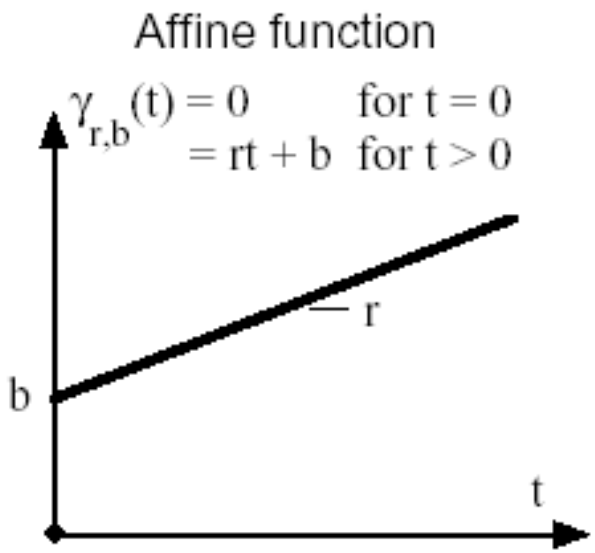
Motivation

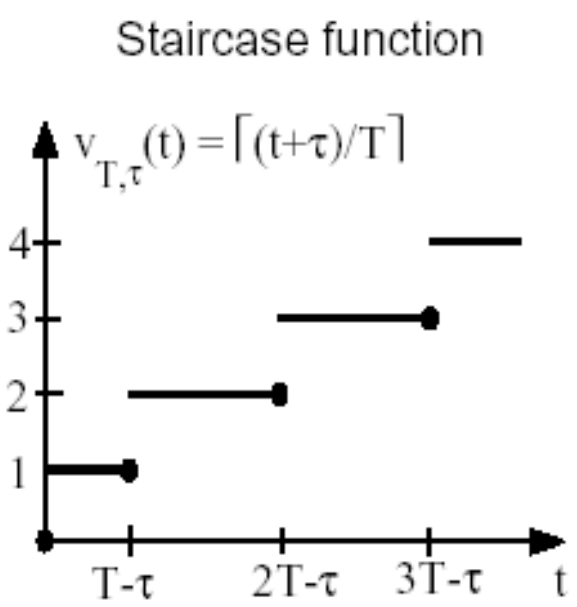
- Why is arrival curve important?

Arrival Curve's Definition

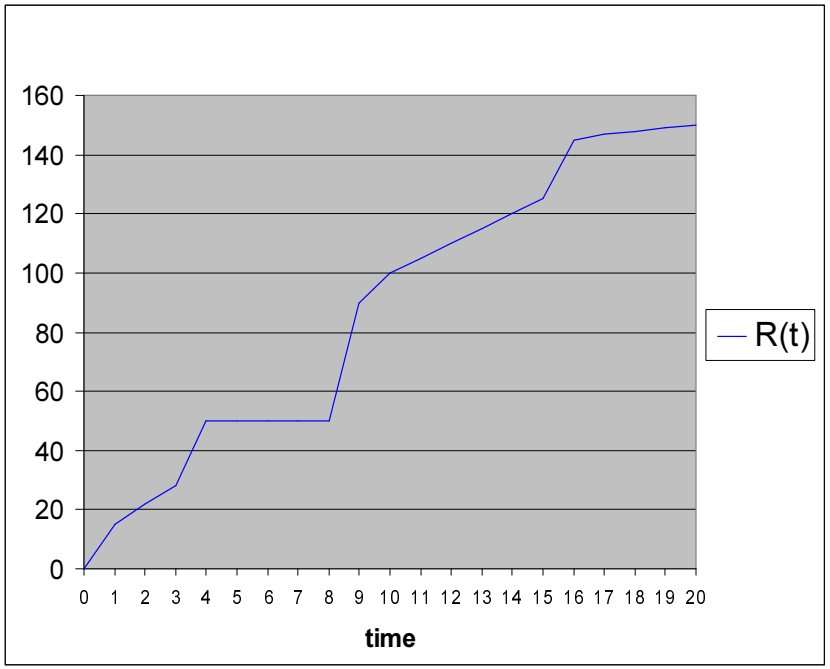
- Function *f* is wide-sense increasing if and only if $f(s) \leq f(t)$ for all $s \leq t$, *We use* $\mathbf{F}$ *to denote the set of wide-sense increasing functions such that* $f(t)=0$ *for* $t \leq 0$.
- Given a wide-sense increasing function α defined for $t \geq 0$ (namely $\alpha \in \mathbf{F}$), we say that a flow *R* is constrained by α if and only if for all $s \leq t$:
- $$R(t)-R(s) \leq \alpha(t-s)$$
- We say that R has α as an ***arrival curve***, or also *R* is α-*smooth*.

Arrival Curve Examples

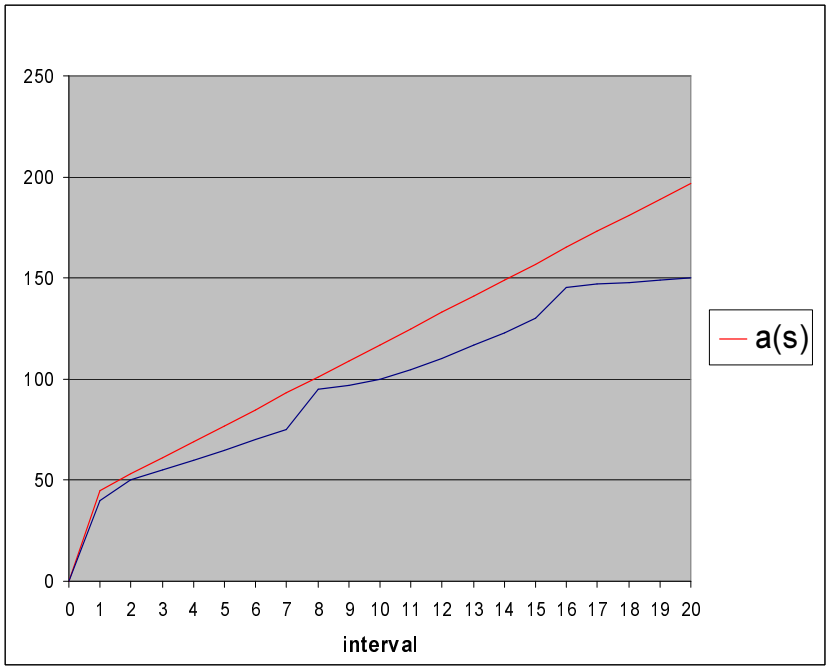




Physical Meaning



Input function R(t)



Arrival curve $\alpha(s)$

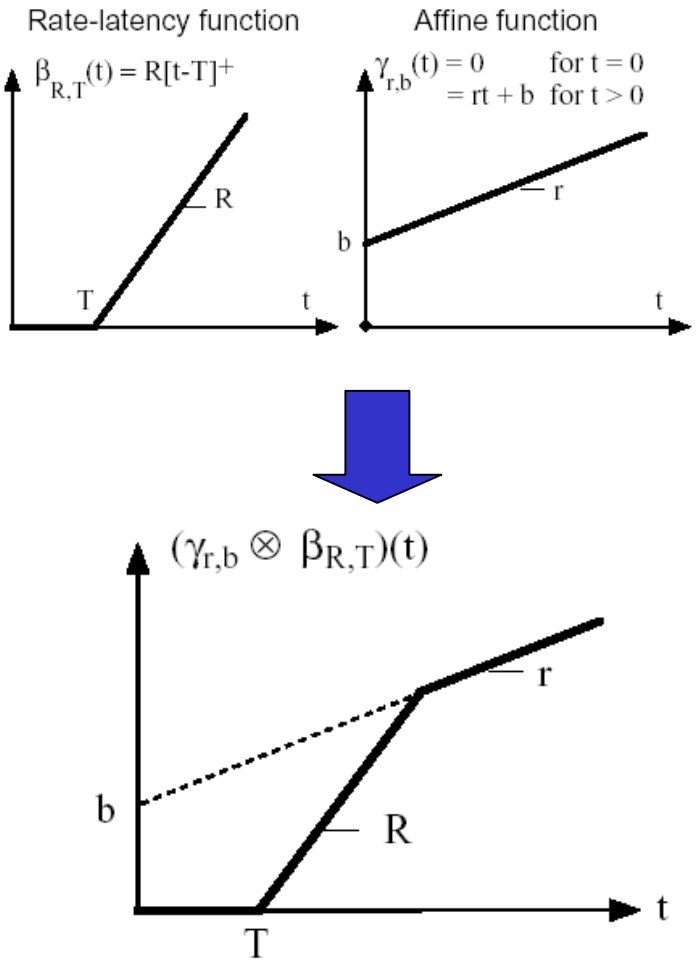
Pause I - What we get now?

- Input and output flow R, R^*
- Backlog and virtue delay
- Arrival curve

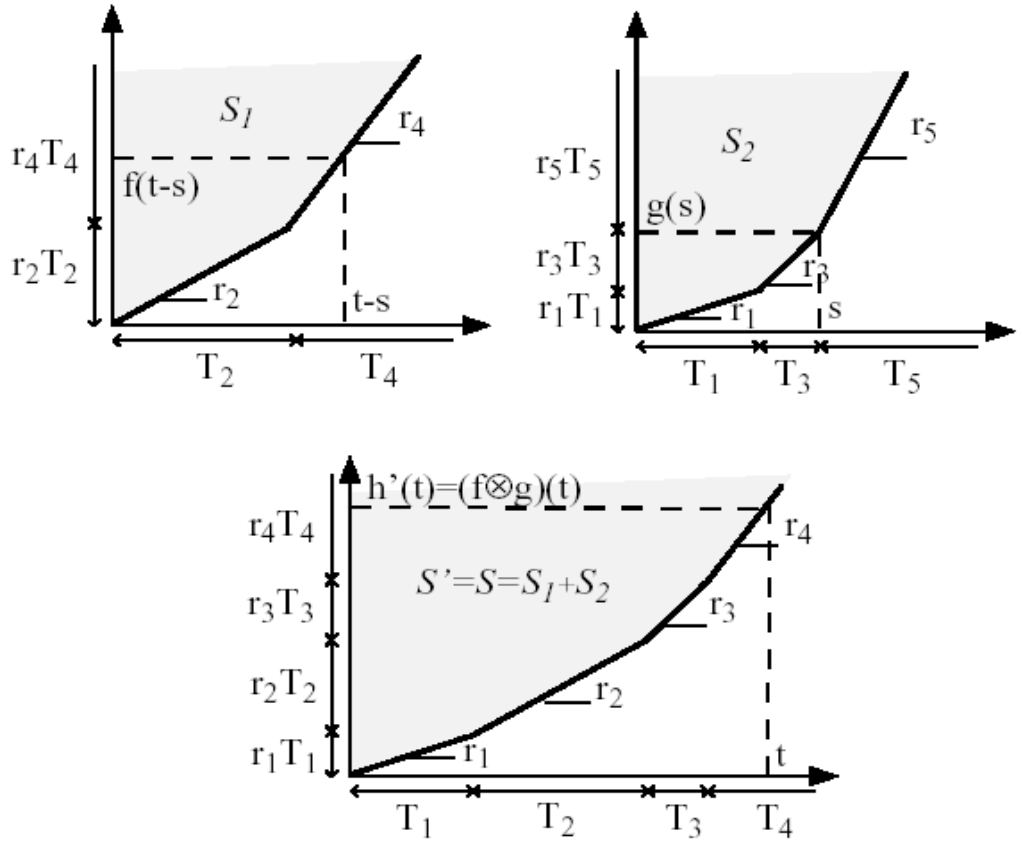
Min-Plus Convolution

- Let f and g be two functions. The min-plus convolution of f and g is the function
  - $(f \otimes g)(t) = inf_{0 \le s \le t} \{ f(t-s) + g(s) \}$
- Properties
  - Commutativity $\alpha \otimes \beta = \beta \otimes \alpha$
  - Associativity $(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$
  - Functions passing through the origin
    - If $f(0)=g(0)=0$ then $f \otimes g \le min(f, g)$
  - Convex property

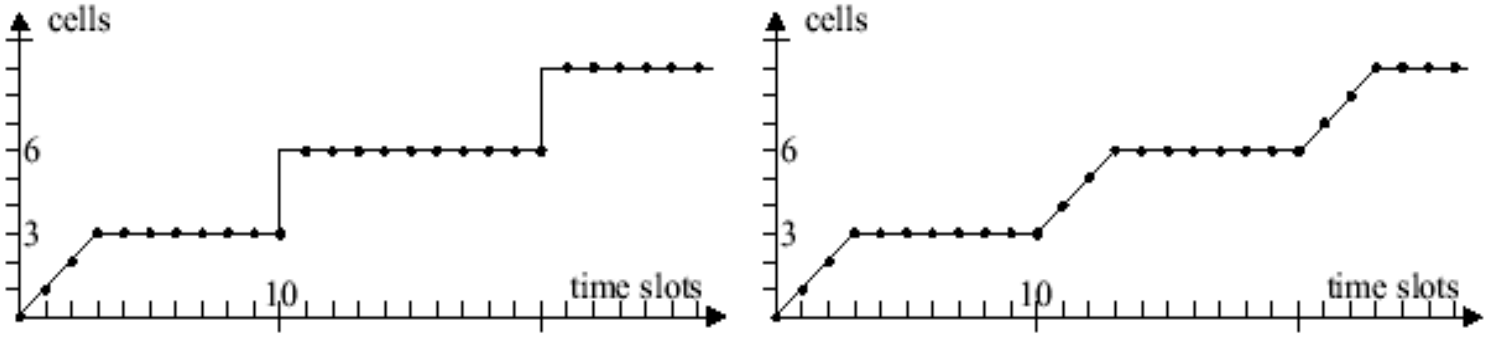
Example


Rate-latency function
$\beta_{R,T}(t) = R[t-T]^+$
R
T
t
Affine function
$\gamma_{r,b}(t) = 0$ for $t = 0$
$= rt + b$ for $t > 0$
r
b
t
$(\gamma_{r,b} \otimes \beta_{R,T})(t)$
r
b
R
t
T

Convex Property



How good is an arrival curve



Good Function

- Function f is sub-additive if and only if $f(t+s) \leq f(t)+f(s)$ for all $s,t \geq 0$
- *Good function*
  - $\alpha(0)=0$ *and* α *is sub-additive*
  - $\alpha = \alpha \otimes \alpha$

How to get a good function

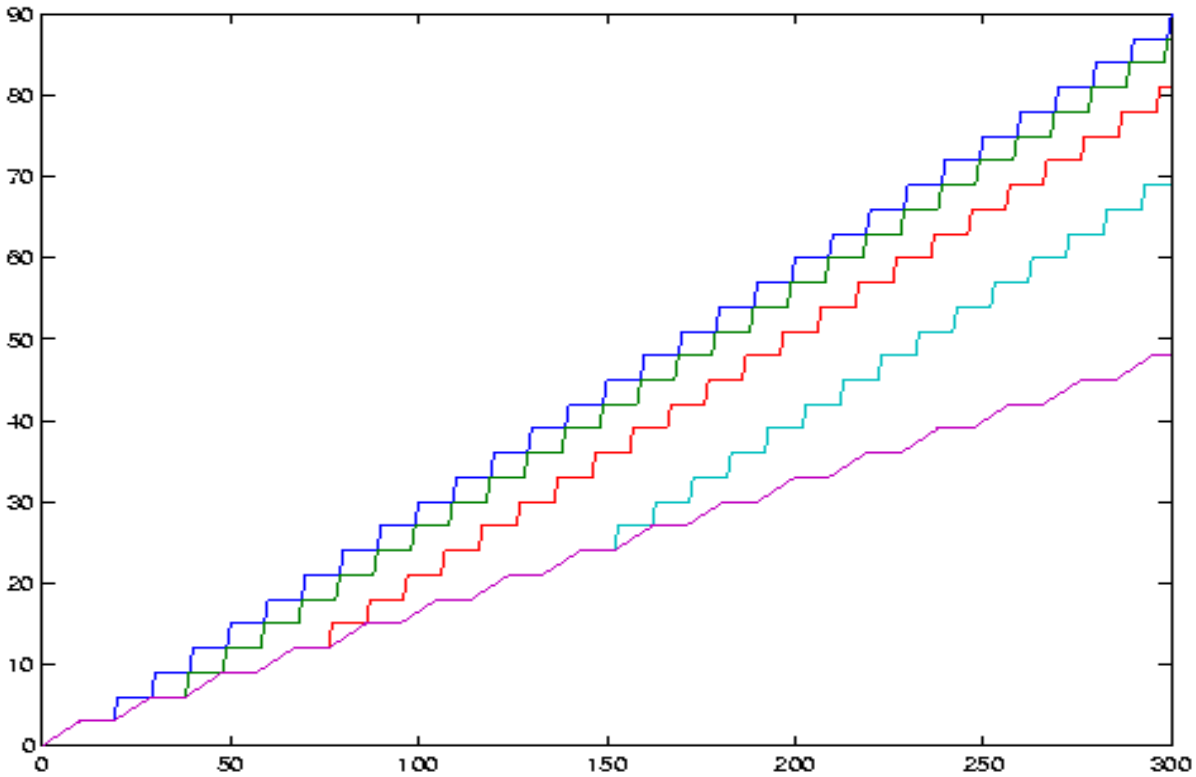
- Reduce to sub-additive function
  - A flow R is constrained by α if and only if

    $$R \le R \otimes \alpha \text{ (Lemma 1)}$$

  - If α_1 *and* α_2 *both are arrival curve then* $\alpha_1 \otimes \alpha_2$ *is also arrival curve (Lemma 2)*
  - Sub-additive closure
    - $0 \le f \otimes f \le f$

      if f is wide-sense increasing and $f(0)=0$ ($f \in \mathbf{F}$)

Matlab Result



original function f f1=f⊗f f2=f1⊗f f3=f2⊗f f4=f3⊗f

Pause II - What we get now?

- Min-plus convolution
- Tighter bound of arrival curve

Service curve

- Definition
- Samples

Service Curve's Definition

- *Consider a system S and a flow through S with input and output function R and R^*. We say that S offers to the flow a* ***service curve*** *β if and only if $\beta \in F$ and $R^* \geq R \otimes \beta$.*
- Physical meaning
  - $R^*(t) - R^*(t_0) \geq \beta(t-t_0)$
  - $R(t_0) - R^*(t_0) = 0$

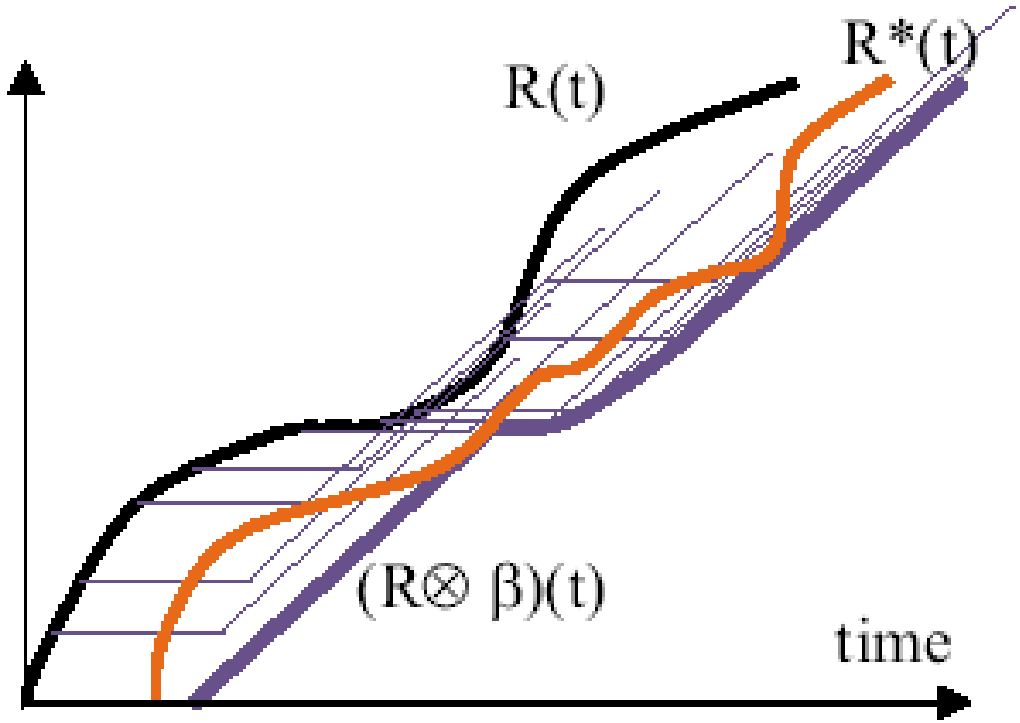
  $\rightarrow R^*(t) \geq R(t_0) + \beta(t-t_0)$

  $\rightarrow R^*(t) \geq inf_{0 \leq s \leq t} [R(s) + \beta(t-s)]$

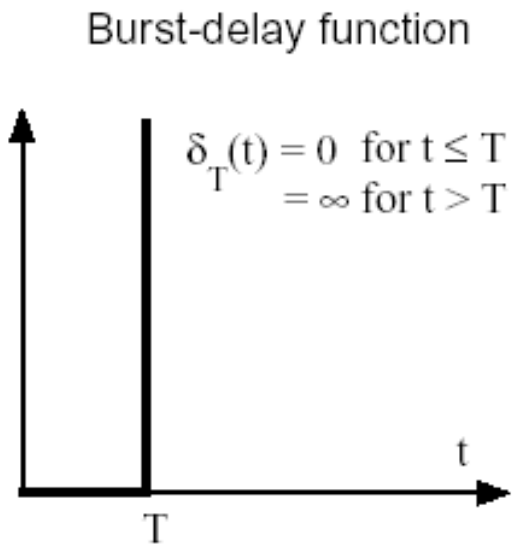
Service Curve's Definition

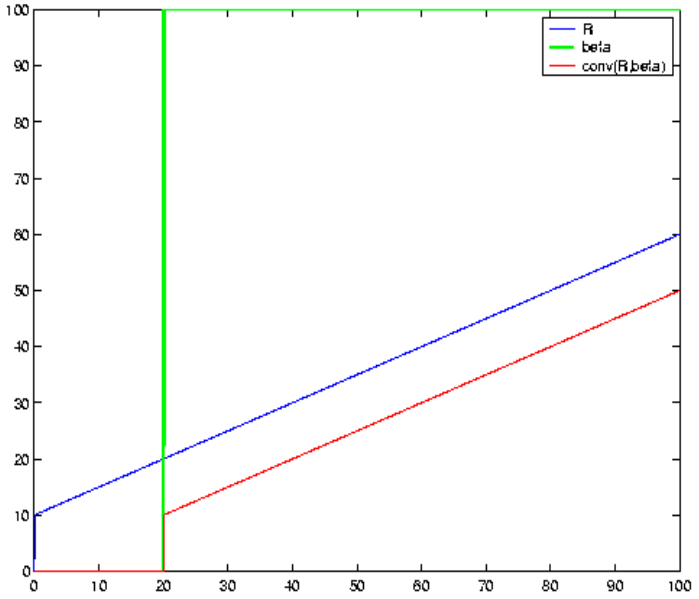
- Intuitive meaning
  - R* 's behavior depends on both R and β
  - Why use convolution?
    - Give freedom to whether starvation exists?
    1. Starvation does not exist, more depends on β
    2. Starvation does exist, more depends on R

Service Curve Example

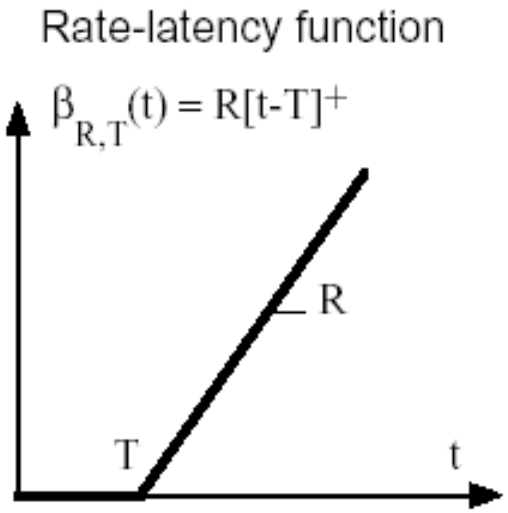


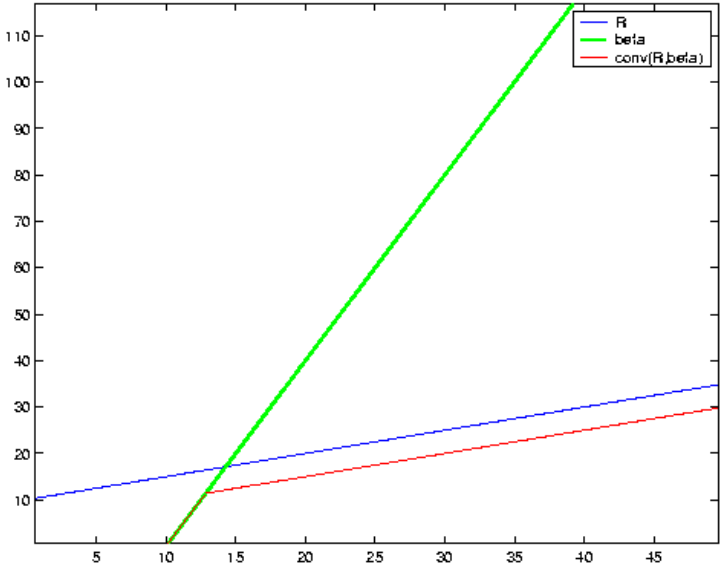
Burst-delay function





Rate-Latency function

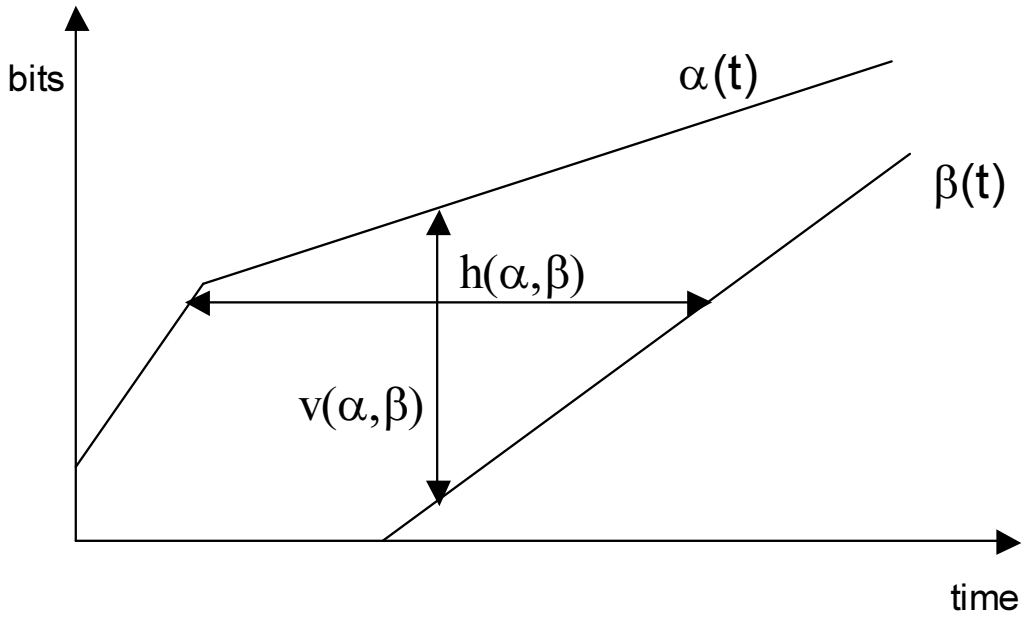




Bounds

- Backlog bound
  - $R(t)-R^{*}(t) \leq \sup_{s \geq 0} \{\alpha(s) - \beta(s)\}$ *(Lemma 3)*
  - *Vertical deviation*
- Delay bound
  - $\delta(s) = inf\{t \geq 0 : \alpha(s) \leq \beta(s+t)\}$
  - h is supremum of all values of δ(s)
  - $d(t) \leq h(\alpha, \beta)$
  - *Horizontal deviation*

Vertical and Horizontal Deviation

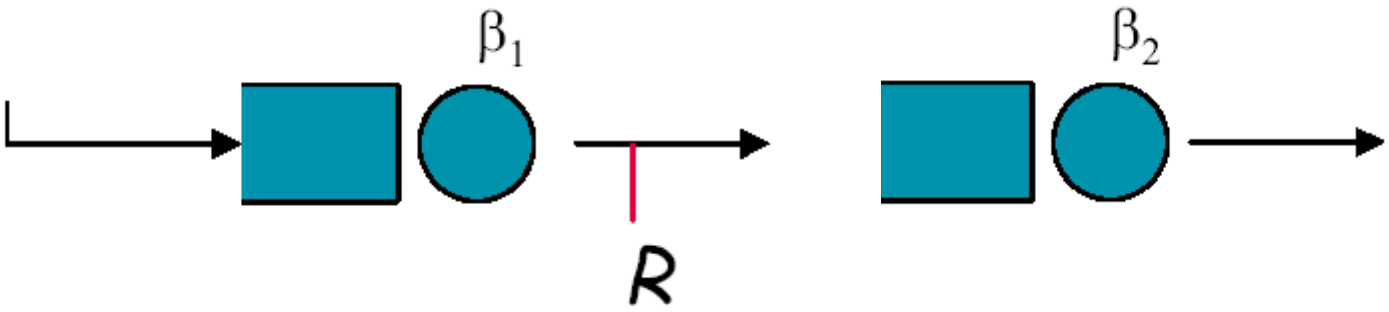


Deconvolution and Output Bound

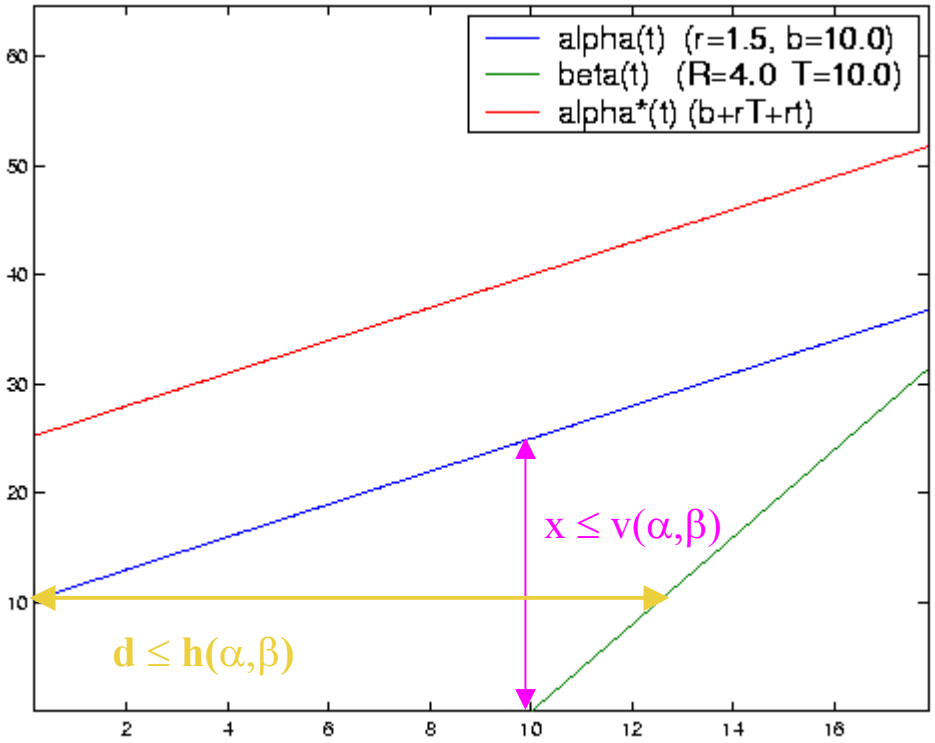
- $f(t) \oslash g(t) = \sup_{u \geq 0} \{f(t+u)-g(u)\}$

- Output flow bound

  $\alpha^* = \alpha \oslash \beta$



Example



Arrival Curve: $\alpha(t) = rt+b = 1.5t+10$

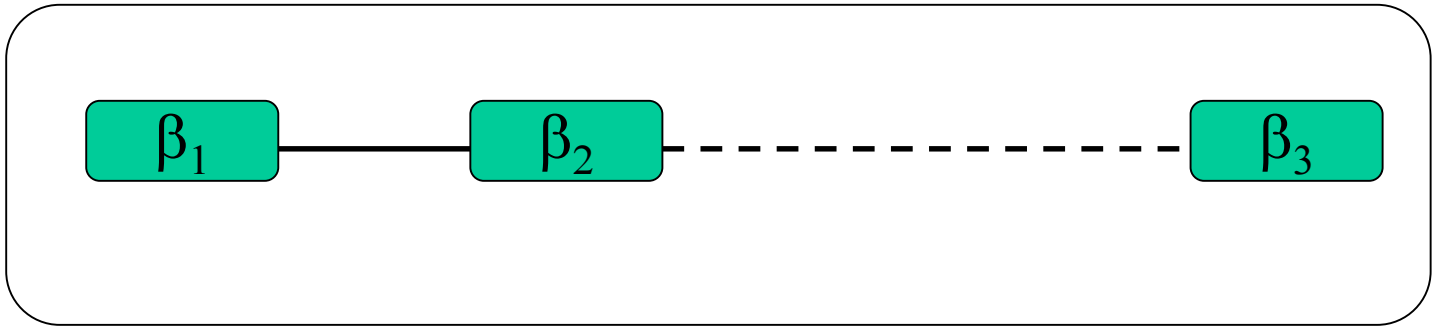
Service Curve: $\beta(t) = R[t-T]^+$

Buffer bound: $x \le v(\alpha,\beta) = b+rT = 10+1.5*10 = 25$

Delay bound: $d \le h(\alpha,\beta) = b/R+T = 10/4.0+10 = 12.5$

Output arrival curve: $\alpha^*(t)=(b+rT)+rt$

Concatenation of Network Elements



$$\beta_1 \otimes \beta_2 \otimes \bullet\bullet\bullet \otimes \beta_i$$

At output of node 1 : $R1 \geq R \otimes \beta_1$

At output of node 2 : $R^* \geq R1 \otimes \beta_2 \geq (R \otimes \beta_1) \otimes \beta_2 = R \otimes (\beta_1 \otimes \beta_2)$

Greedy Shaper

- Shaper with shaping curve δ : a device that forces its output to have δ as arrival curve
- Greedy : delay the input bits in a buffer, wherever sending a bit would violate the constraint δ, but outputs them as soon as possible.
- For an input flow x, the output is given by :
  - $y = \delta \otimes x$ (stronger than the service curve property)

Some Proof

Lemma 1:

think of the definition of convolution

Lemma 2:

lemma 1 and associativity property

Lemma 3:

$R^*(t) \geq (R \otimes \beta)(t)$

$R(t) - R^*(t) \leq R(t) - (R \otimes \beta)(t)$

$= R(t) - \inf_{0 \leq s \leq t} [R(t-s) + \beta(s)]$

$= \sup_{0 \leq s \leq t} [R(t) - R(t-s) - \beta(s)]$

$= \sup_{0 \leq s \leq t} [\alpha(s) - \beta(s)]$