# Real Analysis 24-11-08

## Chap 4 Lebesgue spaces.

### § 4.3 Lebesgue spaces.

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $p > 0$.

A measurable function $f$ on $X$ is said to be

p-integrable if

$$\int |f|^p \, d\mu < \infty.$$

Moreover, we write

$$\|f\|_p := \left( \int |f|^p \, d\mu \right)^{1/p}$$

We call it the p-norm of $f$.

Prop 4.10 (Hölder inequality)

Let $1<p<\infty$. Then

$$\int |fg|\,d\mu \leq \left(\int |f|^p d\mu\right)^{1/p} \cdot \left(\int |g|^q d\mu\right)^{1/q},$$

where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Prop 4.11 (Minkowski inequality)

Let $p \geq 1$. Then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

The proof of the above propositions is based on the following

(Young's inequality)

Let $\alpha, \beta \geq 0$. Let $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$

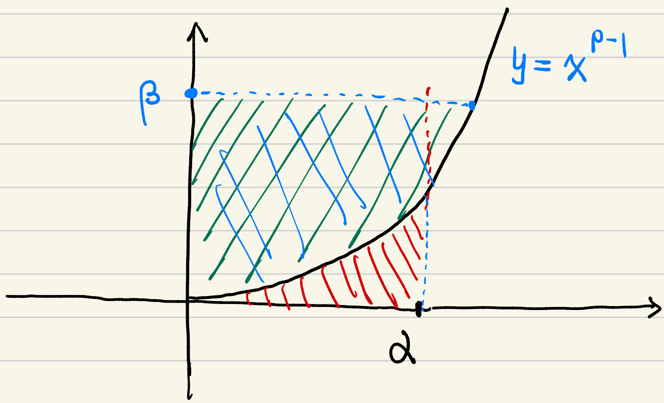
Then $$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

"=" holds $\Leftrightarrow$ $\beta = \alpha^{p-1}$.

Pf of Young's inequality:

We use a geometric approach. Consider the function $y = x^{p-1}$. Its inverse is

$x = y^{q-1}$ (using $(p-1)(q-1)=1$)




Area of the red shaded region

$$= \int_0^\alpha x^{p-1}\,dx = \frac{x^p}{p}\Big|_0^\alpha = \frac{\alpha^p}{p}$$

Area of the blue shaded region

$$= \int_0^{\beta} y^{q-1}\,dy = \frac{\beta^q}{q}$$

From the above geometry, we see that

$$\alpha\beta \le \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

Clearly "=" holds $\Leftrightarrow$ $\beta = \alpha^{p-1}$.

$\boxed{//}$

Proof of the Hölder inequality:

Let $p>1$. Let $f, g$ be measurable functions on $X$. WLOG, we may assume $|f|, |g| < \infty$.

Set $\alpha(x) = \dfrac{|f(x)|}{\|f\|_p}$, $\beta(x) = \dfrac{|g(x)|}{\|g\|_q}$,

here $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Using Young's inequality to $\alpha(x)$, $\beta(x)$, we obtain

$$\frac{|f(x)\, g(x)|}{\|f\|_p \, \|g\|_q} \leq \frac{|f(x)|^p}{p \cdot \|f\|_p^p} + \frac{|g(x)|^q}{q \cdot \|g\|_q^q}.$$

Takig integration w.r.t $\mu$, we have

$$\begin{aligned} \frac{1}{\|f\|_p \|g\|_q} \int |fg| \, d\mu &\leq \frac{1}{p} \int \frac{|f(x)|^p}{\|f\|_p^p} \, d\mu(x) \\ &\quad + \frac{1}{q} \int \frac{|g(x)|^q}{\|g\|_q^q} \, d\mu(x) \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

from which we obtain

$$\int |fg| \, d\mu \leq \|f\|_p \cdot \|g\|_q. \qquad \blacksquare$$

## Proof of the Minkowski inequality:

We prove this by applying the Hölder inequality.

If $p=1$, then since

$$|f(x)+g(x)| \leq |f(x)| + |g(x)|,$$

taking integration gives

$$\|f+g\|_1 \leq \|f\|_1 + \|g\|_1 .$$

Next we assume $1<p<\infty$.

$$|f+g|^p \leq |f|\cdot|f+g|^{p-1} + |g|\cdot|f+g|^{p-1}$$

Taking integration gives

$$\|f+g\|_p^p \leq \int |f|\cdot|f+g|^{p-1}d\mu + \int |g|\cdot|f+g|^{p-1}d\mu$$

(Using Hölder)

$$\leq \left(\int |f|^p d\mu\right)^{1/p}\cdot\left(\int |f+g|^{(p-1)q} d\mu\right)^{1/q}$$

$$+\left(\int |g|^p d\mu\right)^{1/p}\cdot\left(\int |f+g|^{(p-1)q} d\mu\right)^{1/q}$$

$$= \|f\|_p \cdot \|f+g\|_p^{p/q}$$

$$+ \|g\|_p \cdot \|f+g\|_p^{p/q} \quad (\text{using } (p-1)q = p)$$

$$= \left(\|f\|_p + \|g\|_p\right) \cdot \|f+g\|_p^{p/q}.$$

Hence $\quad \|f+g\|_p^{p-\frac{p}{q}} \leq \|f\|_p + \|g\|_p.$

Noticing that $p - \frac{p}{q} = 1$, we obtain the desired inequality. $\blacksquare$

Def. Let $p>0$. Set

$$L^p(X, \mathcal{M}, \mu) = \left\{ \text{all } p\text{-integrable functions on } (X, \mathcal{M}, \mu) \right\}$$

For short, we write $L^p(\mu) := L^p(X, \mathcal{M}, \mu)$.

Recall that for $f \in L^p(\mu)$,

$$\|f\|_p = \left( \int |f|^p \, d\mu \right)^{1/p}.$$

If $\|f\|_p = 0$, then $f = 0$ a.e.

Define $f \sim g$ if $f = g$ a.e.

Then this relation "$\sim$" is an equivalence relation.

Now define

$$\tilde{L}^p(\mu) = L^p(\mu) / \sim$$

For $\tilde{f} \in \tilde{L}^p(\mu)$, define

$$\|\tilde{f}\|_p = \|f\|_p \qquad \text{if } \tilde{f} = [f].$$

Then $\tilde{L}^p(\mu)$ becomes a normed vector space.

Thm 4.12. Let $1 \leq p < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\mu)$. Then $\exists\, f \in L^p(\mu)$ such that

$$\| f_n - f \|_p \to 0 \quad \text{as} \quad n \to \infty.$$

As a consequence, $\tilde{L}^p(\mu)$ is a Banach space.

Pf. Since $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence, for any $j \in \mathbb{N}$, $\exists\, n_j \in \mathbb{N}$ such that

$$① \quad \| f_n - f_m \|_p < 2^{-j} \quad \text{if} \quad n, m \geq n_j.$$

We may further require that

$$n_{j+1} > n_j, \qquad j = 1, 2, \cdots$$

By removing a subset of zero measure, we may assume $|f_n(x)| < \infty \quad \forall\, x \in X,\ n \in \mathbb{N}$.

Define

$$g_k(x) = \sum_{j=1}^{k} \left| f_{n_{j+1}}(x) - f_{n_j}(x) \right| .$$

$$g(x) = \sum_{j=1}^{\infty} \left| f_{n_{j+1}}(x) - f_{n_j}(x) \right| .$$

Clearly $g(x) = \lim_{k \to \infty} g_k(x)$.

Using the Minkowski inequality to $g_k$ gives

$$\|g_k\|_p \leq \sum_{j=1}^{k} \| f_{n_{j+1}} - f_{n_j} \|_p$$

$$\leq \sum_{j=1}^{k} 2^{-j} \qquad (\text{by } (1)).$$

$$< 1$$

By Fatou's lemma,

$$\|g\|_p^p = \int |g(x)|^p d\mu(x) = \int \lim_{k \to \infty} |g_k(x)|^p d\mu(x)$$

$$\leq \underline{\lim}_{k \to \infty} \int |g_k(x)|^p d\mu(x)$$

$$\leq 1$$

Hence $g(x) < \infty$ for $\mu$-a.e $x$.

That is, for $\mu$-a.e $x$,

$$\sum_{j=1}^{\infty} \left| f_{n_{j+1}}(x) - f_{n_j}(x) \right| < \infty$$

Consider the sum

$$(2) \quad f_{n_1}(x) + \sum_{j=1}^{\infty} \left( f_{n_{j+1}}(x) - f_{n_j}(x) \right),$$

which converges for $\mu$-a.e. $x$.

Let $f(x)$ be the above sum if (2) converges otherwise, let $f(x) = 0$.

Then for $\mu$-a.e $x \in X$,

$$f(x) = \lim_{k\to\infty}\left(f_{n_1}(x) + \sum_{j=1}^{k}\left(f_{n_{j+1}}(x) - f_{n_j}(x)\right)\right)$$

$$= \lim_{k\to\infty} f_{n_{k+1}}(x).$$

That is, $\quad f_{n_k} \to f$ a.e.

In what follows we prove that

$$\|f_n - f\|_p \to 0 \quad \text{as} \quad n \to \infty.$$

Let $\varepsilon > 0$. Take a large $N \in \mathbb{N}$ so that

(3) $\quad \|f_n - f_m\|_p < \varepsilon \quad \forall\ n, m > N$

For any $m > N$, by Fatou's lemma,

$$\|f - f_m\|_p^p = \int |f-f_m|^p \, d\mu$$

$$= \int \varliminf_{j\to\infty} |f_{n_j} - f_m|^p \, d\mu$$

$$\leq \varliminf_{j\to\infty} \int |f_{n_j} - f_m|^p \, d\mu$$

$$\leq \varepsilon^p$$

That is, $\|f - f_m\|_p < \varepsilon$.

So $\|f\|_p \leq \|f - f_m\|_p + \|f_m\|_p$

$< \infty$

and $\|f_n - f\|_p \to 0$ as $n \to \infty$.

$\boxtimes$

Recall that a simple function on $(X, \mathcal{M}, \mu)$ is of the form

$$s = \sum_{j=1}^{n} \alpha_j \chi_{E_j}(x),$$

where $\alpha_j \in \mathbb{R} \setminus \{0\}$, $E_j \in \mathcal{M}$.

Let $\mathcal{S} = \{ s : s$ is a simple function $\sum_{j=1}^{n} \alpha_j \chi_{E_j}$ with $\mu(E_j) < \infty \}$.

Prop 4.14. Let $p \geq 1$. Then

$\mathcal{S}$ is dense in $\underline{L^p(\mu)}$.

Pf. Clearly $\mathcal{S} \subset L^p(\mu)$.

Next assume $f \in L^p(\mu)$, $f$ is non-negative.

Then $\exists$ a sequence $(s_k)_{k=1}^{\infty}$ of non-negative simple functions, $s_k \uparrow f$.

Then by Lebesgue Dominated Convergence Thm,

$$\int |s_k - f|^p \, d\mu \to 0 \quad \text{as} \quad k \to \infty.$$

It follows that

$$\|S_k - f\|_p \to 0 \quad \text{as } k \to \infty.$$

Moreover, when $k$ is large enough, $\|S_k\|_p < \infty$.

Writing $S_k = \sum_{j=1}^{n} \alpha_j \chi_{E_j}$, then

$$\alpha_j^p \mu(E_j) \leq \int |S_k|^p \, d\mu < \infty$$

$$\Rightarrow \mu(E_j) < \infty \Rightarrow S_k \in S$$

In the general case, we write

$$f = f^+ - f^-.$$

Applying the above analysis to $f^+$ and $f^-$, we see that

$$\exists \ (S_k) \subset S \quad \text{s.t} \quad \|S_k - f\|_p \to 0.$$

$\square$

Prop 4.15. Let $X$ be a LCHS, let $\mu$ be a Riesz measure. Then

$$C_c(X) \text{ is dense in } L^p(\mu) \text{ for all } 1 \le p < \infty.$$

Pf. Let $1 \le p < \infty$. By Prop 4.14, it suffices to proved that for given $E \in \mathcal{M}$ with $\mu(E) < \infty$, and $\varepsilon > 0$, $\exists$ $\varphi \in C_c(X)$ such that

$$\| \varphi - \chi_E \|_p < \varepsilon.$$

To show the above result, fix $E \in \mathcal{M}$ with $\mu(E) < \infty$, fix $\varepsilon > 0$. By Lusin's Thm, $\exists$ $\varphi \in C_c(X)$ such that $\|\varphi\|_\infty := \sup_X |\varphi(x)| \le 1$

and

$$\mu\{x : \varphi(x) \ne \chi_E(x)\} < 2^{-p} \varepsilon^p.$$

Then $\| \varphi - \chi_E \|_p^p = \int |\varphi(x) - \chi_E(x)|^p \, d\mu$

$$= \int_{\{x:\ \varphi(x) \neq \chi_E(x)\}} |\varphi - \chi_E|^p \, d\mu$$

$$\leq 2^p \cdot \mu\{x : \varphi(x) \neq \chi_E(x)\}$$

$$< \varepsilon^p,$$

which implies

$$\|\varphi - \chi_E\|_p < \varepsilon. \qquad \boxed{///}$$

Prop 4.16. $L^p(\mathbb{R}^d)$ is separable for $1 \leq p < \infty$.

$\left( := L^p(\mathcal{L}^d), \ \mathcal{L}^d - \text{Lebesgue measure on } \mathbb{R}^d \right)$

Recall that we say a topological space $X$ is separable if $\exists$ a countable subset of $X$ which is dense in $X$.

Pf. Let $B_n = \{x \in \mathbb{R}^d : |x| \leq n\}$, $n \in \mathbb{N}$.

Let $\mathcal{P}_n$ denote the collection of the restriction of polynomials with rational coefficients on $B_n$.

That is, any element $f$ of $\mathcal{P}_n$ is of the form

$$f = \chi_{B_n} \cdot g$$

where $g$ is a polynomial with rational coeffients defined on $\mathbb{R}^d$.

Hence $\mathcal{P}_n$ is countable.

Let $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$. $\mathcal{P}$ is countable.

We show below $\mathcal{P}$ is dense in $L^p(\mathbb{R}^d)$.

By Prop 4.15, $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

It is enough to show that for $\varphi \in C_c(X)$, and $\varepsilon > 0$, $\exists\ h \in \mathcal{P}$ s.t

$$\|\varphi - h\|_p < \varepsilon.$$

Since $\operatorname{spt}(\varphi)$ is compact, $\exists\ n \in \mathbb{N}$ such that

$$\operatorname{spt}(\varphi) \subset B_n := \{x : \|x\| < n\}.$$

Then by Weierstrass approximation Thm,

$\exists\ h \in \mathcal{P}_n$ such that

$$\sup_{x \in B_n} |\varphi(x) - h(x)| < \varepsilon \cdot (\lambda^d(B_n))^{1/p}.$$

Now

$$\|\varphi - h\|_p^p = \int |\varphi(x) - h(x)|^p \, d\mathcal{L}^d(x)$$

$$= \int_{B_n} |\varphi(x) - h(x)|^p \, d\mathcal{L}^d(x)$$

$$\leq \mathcal{L}^d(B_n) \left( \sup_{x \in B_n} |\varphi(x) - h(x)| \right)^p$$

$$\leq \mathcal{L}^d(B_n) \left( \varepsilon \cdot \left( \mathcal{L}^d(B_n) \right)^{1/p} \right)^p$$

$$\leq \varepsilon^p$$

So $\|\varphi - h\|_p < \varepsilon$ . $\square$