Real Analysis 24-10-4

Let us first recall the Riesz representation thm and the construction of the Riesz measure.

Thm (Riesz Representation Thm)

Let X be a LCHS. Let Λ be a positive linear functional on $C_c(X)$. Then $\exists$ a Borel measure μ such that μ is finite on compact sets and

$$\Lambda(f) = \int_X f \, d\mu, \quad \forall f \in C_c(X).$$

Construction of μ:

For any open $G \subset X$,

$$\mu_0(G) = \begin{cases} \sup\{ \Lambda(f) : f < G \} & \text{if } G \neq \emptyset, \\ 0 & \text{if } G = \emptyset. \end{cases}$$

Then for each $E \subset X$, define.

$$\mu(E) = \inf\{ \mu_0(G) : G \supset E, G \text{ is open} \}.$$

Now we will finish the proof of the Riesz representation Thm.

Step 4. $\Lambda(f) = \int f \, d\mu$, $\forall f \in C_c(X)$.

It suffices to show that

$$\Lambda(f) \leq \int f \, d\mu \quad \text{for all } f \in C_c(X).$$

Because if this is true, then the reverse inequality follows by replacing f by $-f$.

Fix $f \in C_c(X)$. Suppose $f(X) \subset [a, b]$.

Let $\varepsilon > 0$. Pick

$$y_0 < a < y_1 < \cdots < y_n = b$$

such that

$$y_{i+1} - y_i < \varepsilon.$$

Let $K = \operatorname{supp}(f)$. Set

$$E_j = f^{-1}(y_{j-1}, y_j] \cap K.$$

Then

$$K = \bigcup_{j=1}^{n} E_j,$$ with union being disjoint.

Since K is compact, $\mu(K) < \infty$ so $\mu(E_j) < \infty$.

Choosing open G_j such that

① $\mu(E_j) \geq \mu(G_j) - \frac{\varepsilon}{n}$

② $y_{j-1} - \varepsilon < f(x) < y_j + \varepsilon$, $\forall x \in G_j$.

As $K \subset \bigcup_{j=1}^{n} G_j$, $\exists$ $\varphi_j \prec G_j$ with

$$\sum_{j=1}^{n} \varphi_j = 1 \quad \text{on } K.$$

Hence

$$f = \sum_{j=1}^{n} f \varphi_j .$$

So

$$\begin{aligned}
\Lambda(f) &= \sum_{j=1}^{n} \Lambda(f \varphi_j) \\
&\leq \sum_{j=1}^{n} \Lambda\left((y_j + \varepsilon) \varphi_j\right) \\
&= \sum_{j=1}^{n} (y_j + \varepsilon) \Lambda(\varphi_j) \\
&= \sum_{j=1}^{n} \left(|a| + y_j + \varepsilon\right) \Lambda(\varphi_j) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j) \\
&< \sum_{j=1}^{n} \left(|a| + y_j + \varepsilon\right) \mu(G_j) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)
\end{aligned}$$

(since $|a| + y_j + \varepsilon > 0$)

$$\begin{aligned}
&\leq \sum_{j=1}^{n} \left(|a| + y_{j-1} + 2\varepsilon\right) \cdot \left(\mu(E_j) + \frac{\varepsilon}{n}\right) \\
&\qquad - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)
\end{aligned}$$

$$\leq \sum_{j=1}^{n} y_{j-1}\mu(E_j) + |a|\cdot\left(\sum_{j=1}^{n}\mu(E_j) - \sum_{j=1}^{n}\Lambda(\varphi_j)\right) + O(\varepsilon).$$

$$\leq \int f\, d\mu + O(\varepsilon).$$

$\mu(K) \leq \sum_{j=1}^{n}\Lambda(\varphi_j)$

(since $\sum_j y_{j-1}\chi_{E_j} \leq f$.)

(Here we used $\mu(K) \leq \Lambda(\sum \varphi_j)$ by Step 3).

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Regularity of Riesz measures

Def. Let μ be a Borel measure on a topological space X. A set $E \subset X$ is said to be outer regular if

$$\mu(E) = \inf \{ \mu(G) : G \text{ is open}, G \supset E \}$$

We say that E is inner regular if

$$\mu(E) = \sup \{ \mu(K) : K \text{ is compact}, K \subset E \}.$$

- Moreover, we say μ is regular if

all measurable sets in X is both outer and inner regular with respect to μ.

Prop 2.9. Let Λ be a positive linear functional on a LCHS X. Let $\mu = \mu_\Lambda$ be the Riesz measure associated with Λ. Then the following hold:

(1) Every set in X is outer regular.

(2) Every open set in X is inner regular.

(3) Every measurable set with finite measure is inner regular.

Pf. (1) follows from the definition of μ.

Next we prove (2). Let $G \subset X$ be open. Recall that

$$\begin{aligned}\mu(G) &= \sup\{\Lambda(f) : f \prec G\} \\ &= \sup\left\{\int_X f\,d\mu : f \prec G\right\} \\ &\leq \sup\{\mu(K) : K \text{ compact}, K \subset G\}\end{aligned}$$

(Reason: for given $f \prec G$, take $K = \mathrm{supp}(f)$.

then

$$\int_X f \, d\mu \leq \mu(K) \quad \text{since } f \leq \chi_K \Big)$$

Now we prove (3), i.e every measurable set of finite measure is inner regular.

Let $A \subset X$ be measurable and $\mu(A) < \infty$.

Let $\varepsilon > 0$. Pick an open G such that $G \supset A$ and

$$\mu(G) \leq \mu(A) + \varepsilon.$$

By the additivity of μ, we obtain

$$\mu(G \setminus A) = \mu(G) - \mu(A) < \varepsilon$$

(we used the assumption $\mu(A) < \infty$)

Pick another open $G_1 \supset G \setminus A$ such that

$$\mu(G_1) \leq \mu(G\setminus A) + \varepsilon.$$

Since $\mu(G\setminus A) = \mu(G) - \mu(A)$, the above inequality implies that

$$\begin{aligned} \mu(A) &\leq \mu(G) - \mu(G_1) + \varepsilon \\ &\leq \mu(G\setminus G_1) + \varepsilon, \end{aligned} \quad (*)$$

where in the second inequality, we use

$$\mu(G\setminus G_1) + \mu(G_1) \geq \mu(G).$$

Since $A \subset G$, it follows that

$$\begin{aligned} A &= G\setminus(G\setminus A) \\ &\supset G\setminus G_1 \qquad (\text{since } G_1 \supset G\setminus A). \end{aligned}$$

Now take a compact set $K \subset G$ such that

$$\mu(G\setminus K) < \varepsilon.$$

Take $\tilde{K} = K \setminus G_1$. Then $\tilde{K}$ is compact, and

$$A \supseteq G \setminus G_1 \supseteq \tilde{K}.$$

Using the fact that $(K \setminus G_1) \cup (G \setminus K) \supset G \setminus G_1$

we obtain

$$\mu(\tilde{K}) = \mu(K \setminus G_1) \geq \mu(G \setminus G_1) - \mu(G \setminus K)$$

$$\geq \mu(A) - \varepsilon - \varepsilon$$

(by (*) and $\mu(G \setminus K) < \varepsilon$)

Hence A is inner regular. This proves (3).

$\blacksquare$

Prop 2.10. Let μ be a Riesz measure on a LCHS X which is σ-finite with respect to μ.

$$\left(\text{i.e. } X = \bigcup_j X_j \text{ with } \mu(X_j) < \infty\right)$$

Then the following hold:

(1) For any measurable $E \subset X$ and $\varepsilon > 0$, there exist an open set G and a closed set F so that

$$F \subseteq E \subset G, \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

(2). For any measurable set $E \subset X$, there exists a G_δ set A and a F_σ set B such that $B \subseteq E \subset A$ and $\mu(A \setminus B) = 0$.

Consequently, $\mathcal{M}_C$ is the completion of $\mathcal{B}_X$.

(3) Every measurable set is inner regular.

Recall that a G_δ set is a countable intersection of open sets ; a F_σ set is a union of countably many closed sets)

pf. Let $E \subset X$ be measurable.

Let $X = \bigcup_j X_j$ with $\mu(X_j) < \infty$.

Write $E_j = X_j \cap E$. Then $\mu(E_j) < \infty$. Now let $\varepsilon > 0$.

Pick open set $G_j \supset E_j$ so that

$$\mu(G_j \setminus E_j) < \varepsilon 2^{-j}.$$

Let $G = \bigcup_{j=1}^{\infty} E_j$.

Then $$G \setminus E = \left(\bigcup_j G_j\right) \setminus E$$
$$\subset \bigcup_j \left(G_j \setminus E_j\right)$$

Hence

$$\mu(G \setminus E) \leq \sum_j \mu(G_j \setminus E_j) < \sum_j \varepsilon \cdot 2^{-j} = \varepsilon.$$

Using a similar argument for E^c, we can find an open $G_1 \supset E^c$ such that

$$\mu\left(G_1 \setminus (E^c)\right) < \varepsilon.$$

Notice that $$G_1 \setminus (E^c) = G_1 \cap E$$
$$= E \setminus G_1^c$$

Set $F = G_1^c$. Then F is closed

and

$$E \setminus F = G_1 \setminus E^c$$

Hence $\mu(E \setminus F) < \varepsilon$.

Since $F \subset E \subset G$,

we have

$$\mu(G \setminus F) = \mu(G \setminus E) + \mu(E \setminus F)$$

$< 2\varepsilon$.

(because $G \setminus F = (G \setminus E) \cup (E \setminus F)$)

This proves (1).

Next we prove (2). By (1), we can find open sets (G_n), closed sets (F_n) such that

$$\begin{cases} F_n \subset E \subset G_n, & n \in \mathbb{N} \\ \mu(G_n \setminus F_n) < 2^{-n} \end{cases}$$

Let $$A = \bigcap_n G_n,$$
$$B = \bigcup_n F_n.$$

Then A is a G_δ set and B is a F_σ set.

Clearly $$B \subset E \subset A,$$ and
$$\mu(A \setminus B) \leq \mu(G_n \setminus F_n) < 2^{-n},$$
which implies $\mu(A \setminus B) = 0$.

This proves (2).

Finally we prove (3). It suffices to prove
$$\sup\{\mu(K) : K \text{ compact}, K \subset E\} = \infty$$
if $\mu(E) = +\infty$.

For this, let $X = \bigcup_j X_j$ with $\mu(X_j) < \infty$

Lettig $E_j = E \cap X_j$, we have

$$E = \bigcup_{j=1}^{\infty} E_j$$

Hence $\mu\left(\bigcup_{j=1}^{\infty} E_j \right) = \infty$, which implies

$$\mu\left(\bigcup_{i=1}^{N} E_j \right) \to \infty \quad \text{as } N \to \infty.$$

Now for each N, $\mu\left(\bigcup_{i=1}^{N} E_j \right) < \infty$ we can find a compact $K_N \subset \bigcup_{j=1}^{N} E_j$ with

$$\mu(K_N) \geq \mu\left(\bigcup_{j=1}^{N} E_j \right) - \frac{1}{N}.$$

$$\to +\infty \quad \text{as } N \to \infty.$$

$\square$

§ 2.5. Lusin's Thm.

Thm 2.12. Let μ be a Riesz measure on a LCHS X. Let $f: X \to \mathbb{R}$ be measurable such that f vanishes on A^c for some measurable set A with finite measure. Then for any $\varepsilon > 0$, $\exists$ $g \in C_c(X)$, such that

$$\mu\{x : f(x) \neq g(x)\} < \varepsilon.$$

Pf. Writing $f = f^+ - f^-$, we may simply assume f is non-negative.

Also we may assume f is bounded and A is compact by an approximation argument.

Dividing f by a large number, we may assume $0 \leq f < 1$.

Next we construct simple functions $S_n \uparrow f$.

such that for $n = 1, 2, \cdots,$

$$S_n(x) = \frac{j}{2^n}, \quad \text{if } \frac{j}{2^n} \le f(x) < \frac{j+1}{2^n}$$

for some $j = 0, 1, \cdots, 2^n - 1$.

Notice that $S_1(x) = \frac{1}{2}$ or 0 and

$$S_n(x) - S_{n-1}(x) = \frac{1}{2^n} \text{ or } 0 \text{ for } n \ge 2.$$

Letting $S_0(x) = 0$, then

$$S_n(x) - S_{n-1}(x) = \frac{1}{2^n} \chi_{T_n}(x), \quad n \ge 1,$$

where T_n is the set of points x at which

$$S_n(x) - S_{n-1}(x) = \frac{1}{2^n}.$$

Notice that $T_n \subset A$.

(because on A^c, $f(x) = 0$ so $S_n(x) = 0$)

Next notice that

$$f(x) = \sum_{n=1}^{\infty} S_n(x) - S_{n-1}(x) \Big)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \chi_{T_n}(x).$$

Since A is compact, we can choose an open set V such that

$$A \subset V, \quad \overline{V} \text{ is compact.}$$

Now Let $\varepsilon > 0$. For each n, pick an open set G_n and compact K_n such that

$$K_n \subset T_n \subset G_n \subset V \subset \overline{V},$$

So that $\mu(G_n \setminus K_n) < \frac{\varepsilon}{2^n}$.

By the Urysohn lemma, $\exists$ $h_n \in C_c(X)$

$$K_n \prec h_n \prec G_n$$

Now we define

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n} h_n .$$

Hence $g \in C_c(X)$. (Because $g = 0$ on $\bar{V}^c$)

Notice that

$$h_n = \chi_{T_n} \quad \text{except on the set } G_n \setminus K_n .$$

(Since on K_n, $h_n = \chi_{T_n} = 1$

on G_n^c, $h_n = 0 = \chi_{T_n}$)

Hence $f = g$ except on $\bigcup_n (G_n \setminus K_n)$

It follows that

$$\{x : f(x) \neq g(x)\} \subset \bigcup_n G_n \setminus K_n$$

However,

$$\mu\Big(\bigcup_n G_n \setminus K_n\Big) \leq \sum_n \mu(G_n \setminus K_n)$$

$$< \sum_n \varepsilon \cdot 2^{-n} = \varepsilon.$$

Corollary 2.13.

Under the assumption of Thm 2.12, let f be a measurable function satisfying $|f| \leq 1$.

then $\exists$ a sequence $(g_n) \subset C_c(X)$ such that

$$\lim_{n\to\infty} g_n(x) = f(x) \quad \text{a.e.}$$

Pf. By Lusin's Thm, we can find for $n \in \mathbb{N}$,

$g_n \in C_c(X)$ and $E_n \subset X$ measurable

such that

$$\mu(E_n) < \frac{1}{2^n} \quad \text{and} \quad f = g_n \text{ on } E_n^c.$$

Then

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

By Borel-Cantelli lemma,

$$\mu\{x : x \in E_n \text{ for infinitely many } n\} = 0$$

Hence for almost all point x,

x belongs to finitely many E_n's

and let $n_0(x)$ be the largest such n.

Then

$$g_n(x) = f(x) \quad \text{for all } n \geq n_0(x).$$

Hence $\lim_{n\to\infty} g_n(x) = f(x)$. ▨