Integration- Supplement

See If You Need This Video!

1. Which of the followings corresponds to $\int_1^3 f(x)\,\mathrm{d}x$?

   A. $\lim_{N\to\infty} \sum_{n=1}^{N} f(n/N)(1/N)$

   B. $\lim_{N\to\infty} \sum_{n=1}^{N} f(1+n/N)(1/N)$

   C. $\lim_{N\to\infty} \sum_{n=1}^{N} f(2n/N)(2/N)$

   D. $\lim_{N\to\infty} \sum_{n=1}^{N} f(1+2n/N)(1/N)$

   E. $\lim_{N\to\infty} \sum_{n=1}^{N} f(1+2n/N)(2/N)$

2. If

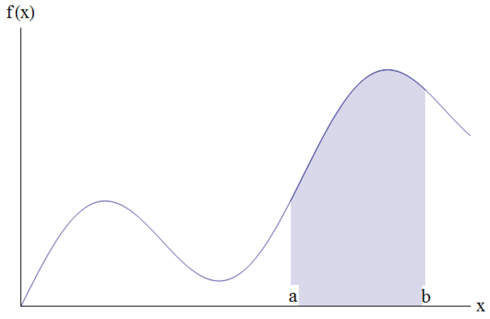
$$\int_a^b f(x)\,\mathrm{d}x = 5$$

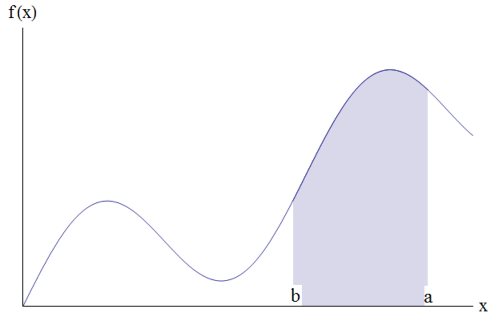
   Which of the followings $= -5$?

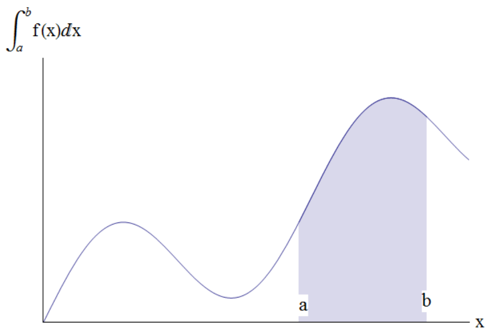
   1. $\int_b^a f(x)\,\mathrm{d}x$
   2. $\int_{-a}^{-b} f(x)\,\mathrm{d}x$
   3. $\int_{-a}^{b} f(x)\,\mathrm{d}x$
   4. $\int_{-a}^{-b} f(-x)\,\mathrm{d}x$

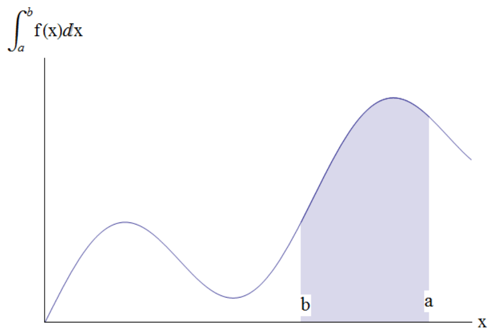
   A. 1 only. B. 2 only. C. 1, 4 only. D. 2, 3 only.
   E. All are correct.

3. Which of the followings highlighted area equals to $\int_a^b f(x)\, \mathrm{d}x$?

A. 

B. 

C. 

D. 

E. Integration does nothing with area.

4. Which of the following(s) is (are) correct? C is a constant.

1. $\int \, \mathrm{d}x = x + C$
2. $\int x^{-3} \, \mathrm{d}x = -x^{-2}/2 + C$
3. $\int x^{-1} \, \mathrm{d}x = -x + C$
4. $\int x^{20} \, \mathrm{d}x = x^{19}/20 + C$

A. 1 only.
B. 4 only.
C. 1, 2 only.
D. 3, 4 only.
E. None of them is correct.

5. Which of the following(s) is (are) correct? C is a constant.

   1. $\int e \, \mathrm{d}x = e + C$
   2. $\int e^x \, \mathrm{d}x = e^x + C$
   3. $\int 2^x \, \mathrm{d}x = 2^x + C$
   4. $\int \ln x \, \mathrm{d}x = x^{-1} + C$

A. 2 only.
B. 4 only.
C. 1, 2 only.
D. 2, 3 only.
E. 2, 4 only.

6. Which of the following(s) is (are) correct?

   1. $\int f(x) \, \mathrm{d}x = \int f(u) \, \mathrm{d}u$
   2. $\int_0^1 e^x \, \mathrm{d}x = e - 1$
   3. $\frac{\mathrm{d}\left[\int f(x) \, \mathrm{d}x\right]}{\mathrm{d}x} = f(x)$
   4. $\int \sin x \, \mathrm{d}x = -\cos x + C$

A. 1 only.
B. 3 only.
C. 2, 4 only.
D. 1, 2, 3 only.
E. All are correct.

7. Solve the following integral.

$$\int x^{-1} + \cos x \, \mathrm{d}x$$

A. $(\ln x)(\sin x) + C$
B. $\ln x + \sin x + C$
C. $-(\ln x)(\sin x) + C$
D. $\ln x - \sin x + C$
E. $\ln x^{-1} + \sin x + C$

8. Solve the following integral.

$$\int x^2 e^{x^3+1}\, \mathrm{d}x$$

A. $e^{x^3} + C$
B. $e^{x^3+1} + C$
C. $e^{x^3+1}/3 + C$
D. $xe^{x^3+1} + C$
E. $x^3/3 + e^{x^3}/3 + C$

9. Solve the following integral.

$$\int \ln x\, \mathrm{d}x$$

A. $x^{-1} + C$
B. $x \ln x - x + C$
C. $\ln x + x + C$
D. $\ln x + C$
E. $x \ln x + C$

10. Solve the following integral.

$$\int_0^{\pi} x \sin x^2\, \mathrm{d}x$$

A. $0.5\,(\cos \pi^2 - 1)$
B. $-0.5\,(\cos \pi^2 - 1)$
C. 1
D. -1
E. 0

Learn More

We now consider a definite integral with two layers.

$$\int_0^2 \int_3^4 xy \; \mathrm{d}x \; \mathrm{d}y$$

It means we first integrate xy with respect to x, then y.
As y does not depend on x, so

$$\int_3^4 xy \; \mathrm{d}x = y\frac{1}{2}\left(16 - 9\right) = \frac{7y}{2}$$

Then

$$\begin{aligned}\int_0^2 \int_3^4 xy \; \mathrm{d}x \; \mathrm{d}y &= \int_0^2 \frac{7y}{2} \; \mathrm{d}y \\ &= 7\end{aligned}$$

Also, you may note that we can exchange the order of integration here

$$\int_0^2 \int_3^4 xy \; \mathrm{d}x \; \mathrm{d}y = \int_3^4 \int_0^2 xy \; \mathrm{d}y \; \mathrm{d}x$$

However, the order of integration cannot be exchanged directly. Take a look at the next example.

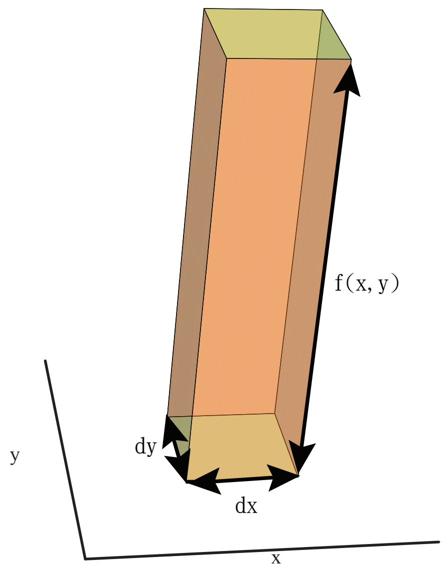
$$\int_0^2 \int_0^y xy \; \mathrm{d}x \; \mathrm{d}y \neq \int_0^y \int_0^2 xy \; \mathrm{d}y \; \mathrm{d}x$$

To answer why, we have to take a look at the meaning of layers of integral.

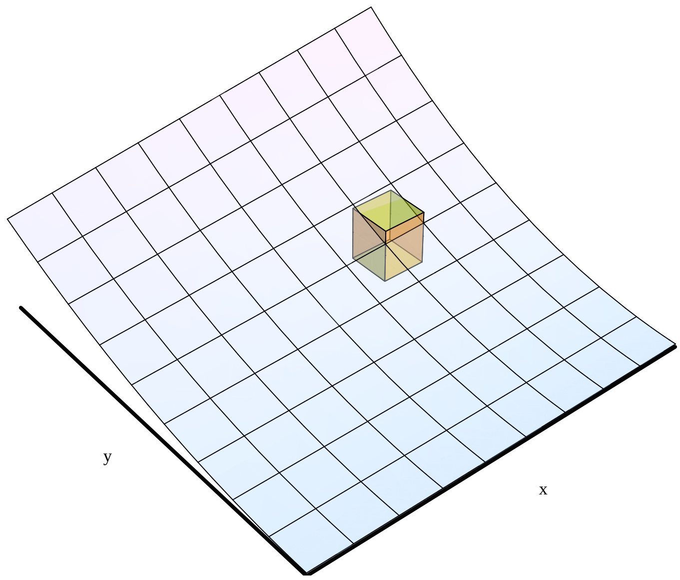
It is a similar idea as area under a curve for one layer integrals. When there are two layers of integrals, it is a volume under a surface. Let's break down the 2-layers integral into elements and compare with the 1-layer integral.

In $\int_{x_1}^{x_2} f\left(x\right) \; \mathrm{d}x$, we are summing area of rectangles with width $\mathrm{d}x$, height $f\left(x\right)$.

In $\int_{y_1}^{y_2} \int_{x_1}^{x_2} f\left(x, y\right) \; \mathrm{d}x \; \mathrm{d}y$, we are summing volume of rectangular parallelepiped with width, depth $\mathrm{d}x$, $\mathrm{d}y$; height $f\left(x, y\right)$. You may take a look at the following figures for a better illustration.



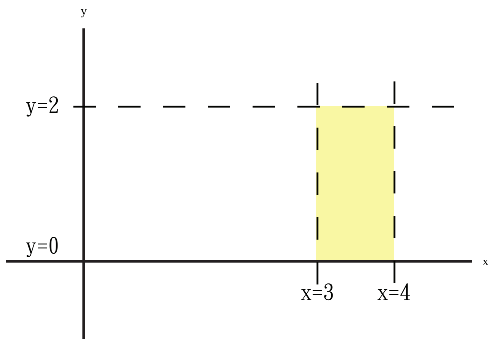
Of course dx and dy are infinitely small. Also, the height $f(x, y)$ of the parallelepiped depends of x, y. (Just like the height $f(x)$ in one layer, it depends on x). Then we are summing the volume of parallelepiped like this to find the volume under a surface $f(x, y)$.



Then get back to our question, why we cannot exchange the order of integration directly? The answer is in the upper and lower limits. Like one layer integration, the upper and lower limits are the range of summation. In

$$\int_0^2 \int_3^4 xy \, \mathrm{d}x \, \mathrm{d}y$$

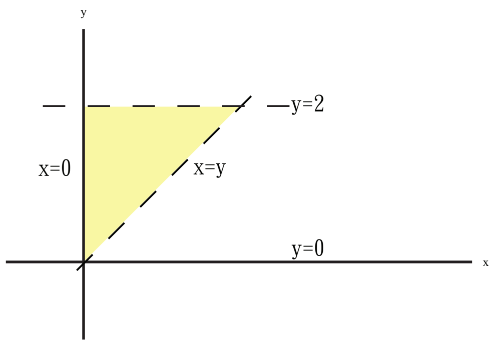
We are summing the volume with $3 < x < 4$ and $0 < y < 2$.



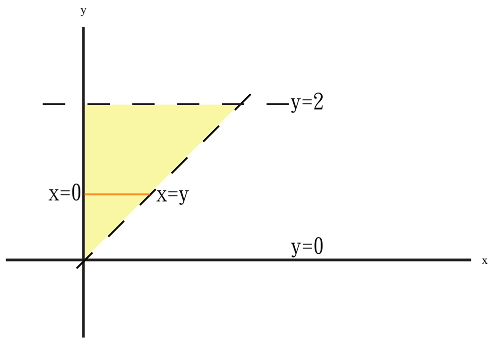
In

$$\int_0^2 \int_0^y xy \, \mathrm{d}x \, \mathrm{d}y$$

We are summing the volume for ranges of x, y like this



Separate the two layers, the inner one $\int_0^y xy \, \mathrm{d}x$ is only working at one y, like this



The the outer layer is summing the result of inner layer with different y.

From this idea, try to figure out why

$$\int_0^2 \int_0^y xy \, \mathrm{d}x \, \mathrm{d}y = \int_0^2 \int_x^2 xy \, \mathrm{d}y \, \mathrm{d}x$$

And Drill Deeper

We have a few challenges for you. If you really have no idea, take a look at the "Guide" and learn the way of thinking. The way to think may help you in solving problems even in real life. And no solution will be given for this part, just enjoy yourself. :)

Challenge 1.
In the video, the examples are not done yet. Try to finish them by yourself.

$$\int \frac{\sin^3 x \cos x}{1+\sin x}\,\mathrm{d}x$$
$$\int x^2 \cos x\,\mathrm{d}x$$

Challenge 2.
You may have to combine techniques, try these

$$\int \frac{e^x}{1-e^{2x}}\,\mathrm{d}x$$
$$\int_0^{\sqrt{\pi}} x e^{x^2} \sin\left(x^2\right)\mathrm{d}x$$

"Guide"

You can always get other ways to solve the problem. This "guide" is just a helping hand, not the law.

Challenge 1.
If you have no idea how to get started, you may first go back to our video and take a look at our attempt first. There is a hint for each of the questions here.

1. $+f(x) - f(x)$ or $\times f(x) / f(x)$ means doing nothing, but it sometimes helps you group or separate functions. The choice of $f(x)$ depends on your goal.

2. You may have to apply an operation (likes substitution or by-parts) more than once, try to foresee if you need from the pattern.

Challenge 2.
When you see a function is "inside" another function, you probably have to start with substitution.