Approximation- Supplement

See If You Need This Video!

1. What is the order of approximation in the following expression?

$$\frac{1}{1-x} \approx 1 + x + x^2 + x^3$$

   A. 2nd order
   B. 3rd order
   C. 4th order
   D. ascending order
   E. well-ordered

2. What is the error in the following approximation?

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

   A. $O\left(x^2\right)$
   B. $O\left(x^3\right)$
   C. $O\left(x^4\right)$
   D. $O\left(x^5\right)$
   E. The error is not presented here.

3. Which of the following is the Taylor series of $f(x)$ about $x = x'$?

A. $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)\, x^n$

B. $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x')\, x^n$

C. $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x')\, x'^n$

D. $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x')\, (x - x')^n$

E. $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x)\, (x - x')^n$

4. What is the range of convergence of the following Taylor series?

$$\frac{1}{\sqrt{6-x}} = \frac{1}{\sqrt{3}} + \frac{(x-3)}{6\sqrt{3}} + \frac{(x-3)^2}{24\sqrt{3}} + \frac{5(x-3)^3}{432\sqrt{3}} + \cdots$$

A. $|x-3| < 1$
B. $|x-3| \leq 1$
C. $|x| < 1$
D. $|x| \leq 1$
E. The series diverges.

5. Which of the following is the the 3rd order approximation of $f_1(x)\, f_2(x)$ around $x = 0$?

$$f_1(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots$$
$$f_2(x) = 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

A. 1
B. $1 + x - \frac{x^3}{3}$
C. $-\frac{x^3}{3}$
D. $2 + x + \frac{x^3}{6}$
E. $\frac{x^3}{6}$

Learn More

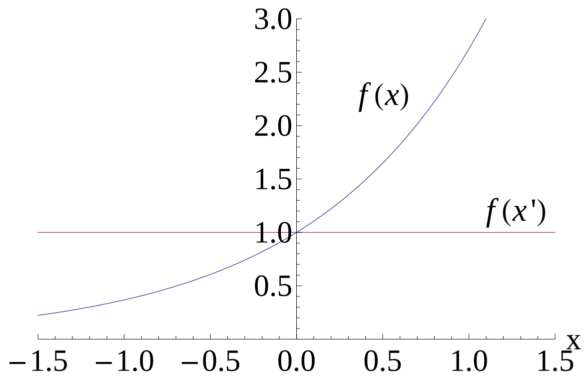
If you observe the Taylor series of a function term by term, you can understand and memorize the Taylor series better.

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}\left(x'\right)\left(x-x'\right)^n$$

When we approximate a function at $x = x'$, it means that we should have the approximation more and more accurate when we x get closer to x'. Alternatively, we may start constructing the approximation from the point $(x', f(x'))$. And that is the 0th order term.

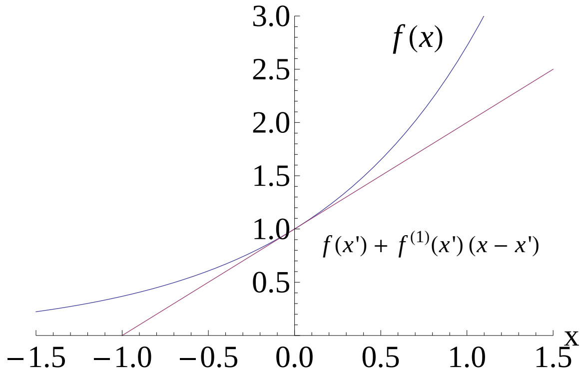
$$f(x) \approx f(x')$$

So 0th order approximation is approximating the whole function $f(x)$ as a constant $f(x')$ as shown below. We take $f(x) = e^x$, $x' = 0$ as an example for plotting, but the same idea can be applied to other functions.



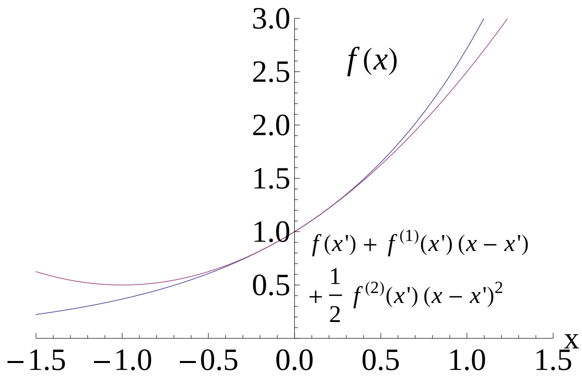
We have a perfect match at exactly $x = x'$, but the 0th order approximation cannot follow the change of $f(x)$. To mimic the change, 1st order term is added. So that the 1st derivative of approximation and $f(x)$ are the same at $x = x'$. (So that they have the same slope at $x = x'$).

$$f(x) \approx f(x') + f^{(1)}(x')(x - x')$$



However, $f(x)$ is curved (the slope $f^{(1)}(x)$ is changing along x). So we have to mimic this change with a curve. Step-by-step, 2nd term is added. So that the 2nd derivative of approximation and $f(x)$ are the same at $x = x'$. (So that they have the same changing rate of slope (1st derivative) at $x = x'$).

$$f(x) \approx f(x') + f^{(1)}(x')(x - x') + \frac{1}{2} f^{(2)}(x')(x - x')^2$$



And with this idea, we match the change of 2nd derivative $f^{(2)}(x')$ with 3rd order term, match the change of 3rd $f^{(3)}(x')$ with 4th order term... And you can see that the Taylor series is actually mimicking how the function $f(x)$ curves.

And Drill Deeper

We have a few challenges for you. If you really have no idea, take a look at the "Guide" and learn the way of thinking. The way to think may help you in solving problems even in real life. And no solution will be given for this part, just enjoy yourself. :)

Challenge 1.
Prove the following typical Taylor series.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$
$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Why is there no x^{2n} term in $\sin x$, no x^{2n+1} term in $\cos x$?

Challenge 2.
Prove the following relations with the above Taylor series.
Just for practice of Taylor series.

1. You may need differentiation here.

$$\frac{\mathrm{d}e^x}{\mathrm{d}x} = e^x$$

2. You may need complex number here.

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

"Guide"

You can always get other ways to solve the problem. This "guide" is just a helping hand, not the law.

Challenge 1.
Expand the LHS function as Taylor series and check if it is consistent with the RHS.
Regarding the last questions, is x^2 an odd function of even function? Further extend this question, what would you say about the function, if its Taylor series has no x^{2n} or x^{2n+1} term?

Challenge 2.
You may expand both sides of the equations or start with one side and group the terms to form the function on the other side.