CSCI3160: Special Exercise Set 11

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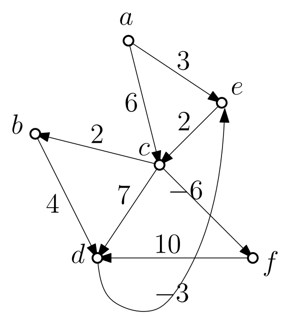
Problem 1. Consider the weighted directed graph $G = (V, E)$ below.



Suppose that we run Johnson's algorithm on G. Recall that the algorithm re-weights all the edges to make sure that every edge should carry a non-negative weight. Give all the edge weights after the re-weighting.

Problem 2 (Textbook Exercise 25.3-4). Recall that, given a weighted directed graph $G = (V, E)$, Johnson's algorithm re-weights all the edges. Prof. Goofy proposes to replace Johnson's re-weighting strategy with the following one:

- Find the smallest edge weight z in G (e.g., for the graph G in Problem 1, $z = -6$).
- Re-weight each edge (u, v) in G by adding $-z$ to its weight, namely, (u, v) carries the weight $w(u, v) - z$ after the re-weighting.

Let G' be the resulting graph obtained by applying Prof. Goofy's strategy. Give an example to show that the strategy does not guarantee the following property: a path π from vertex u to v is a shortest path in G if and only if it is a shortest path in G'.

Problem 3. Let $G = (V, E)$ be a simple directed graph where each edge $(u, v) \in E$ carries a weight $w(u, v)$, which can be negative. Let $h : V \to \mathbb{Z}$ be an arbitrary function (mapping each vertex in V to an integer). For each $(u, v) \in E$, define $w'(u, v) = w(u, v) + h(u) - h(v)$. Let $G' = (V, E)$ be the same graph as G, except that the edges are weighted using w'. Prove: G has a negative cycle if and only if G' does.

Problem 4. Let $G = (V, E)$ be a simple directed graph where $V = \{1, 2, ..., n\}$. The *transitive closure* of G is an $n \times n$ matrix $\boldsymbol{M}$ where

$$\boldsymbol{M}[i, j] = \begin{cases} 1 & \text{if vertex } i \text{ can reach vertex } j \text{ in } G \\ 0 & \text{otherwise} \end{cases}$$

Compute $\boldsymbol{M}$ in $O(|V|(|V| + |E|))$ time.