CSCI3160: Regular Exercise Set 9

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Problem 1*. Prove the correctness of Dijkstra's algorithm (when the edges have non-negative weights).

Problem 2. Consider again your proof for Problem 1. Point out the place that requires edge weights to be non-negative.

Problem 3. Consider a directed simple graph $G = (V, E)$ where each edge $e \in E$ has an arbitrary weight $w(e)$ (which can be negative). It is known that G does not have negative cycles. Prove: given any vertices $s, t \in V$, at least one shortest path from s to t is a simple path (i.e., no vertex appears twice on the path).

Remark: This implies that the path must have at most $|V| - 1$ edges.

Problem 4* (SSSP in a DAG). Consider a simple acyclic directed graph $G = (V, E)$ where each edge $e \in E$ has an arbitrary weight $w(e)$ (which can be negative). Solve the SSSP problem on G in $O(|V| + |E|)$ time.

Problem 5. Let $G = (V, E)$ be a simple directed graph where each edge $e \in E$ carries a weight $w(e)$, which can be negative. It is guaranteed that G has no negative cycles. Prove: given any vertices $s, t \in V$, at least one shortest path from s to t is a simple path (i.e., no vertex appears twice on the path).

Problem 6.** Let $G = (V, E)$ be a simple directed graph where the weight of an edge (u, v) is $w(u, v)$. Prove: the following algorithm correctly decides whether G has a negative cycle.

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algorithm negative-cycle-detection
pick an arbitrary vertex s ∈ V
initialize dist(s) = 0 and dist(v) = ∞ for every other vertex v ∈ V
for i = 1 to |V| − 1
    relax all the edges in E
for each edge (u, v) ∈ E
    if dist(v) > dist(u) + w(u, v) then
        return "there is a negative cycle"
return "no negative cycles"
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Problem 7. Let $G = (V, E)$ be a simple directed graph where every edge (u, v) carries a weight $w(u, v)$, which can be negative. G has no negative cycles. Recall that Johnson's algorithm adds a vertex v_{dummy}, as well as some out-going edges of v_{dummy}, to G and computes the shortest path distance $spdist(v_{dummy}, v)$ from v_{dummy} to every vertex. Then, the weight of each edge (u, v) is modified to:

$$w'(u, v) = w(u, v) + spdist(v_{dummy}, u) - spdist(v_{dummy}, v).$$

Prove: $w'(u, v) \geq 0$.

Problem 8. Let $G = (V, E)$ be a simple directed graph where every edge (u, v) carries a *non-negative* weight $w(u, v)$. Apply Johnson's algorithm to compute a new weight $w'(u, v)$ for each edge $(u, v) \in E$. Prove: $w'(u, v) = w(u, v)$.