ENGG1410-F Tutorial 11

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Problem 1

Calculate

$$\oint_C \boldsymbol{f}(\boldsymbol{r})d\boldsymbol{r}$$

where $\boldsymbol{f} = [y, -x]$, and C is the circle $x^2 + y^2 = 1$ in the positive direction.

Remark: The sign $\oint$ has the same meaning as $\int$ except that the former emphasizes that C is a closed curve.

Problem 2

Define Q as the square in $\mathbb{R}^2$ enclosing all the points (x, y) satisfying $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Calculate $\oint_C \boldsymbol{f}(\boldsymbol{r})d\boldsymbol{r}$, where $\boldsymbol{f} = [6y^2, 2x - 2y^4]$, and C is the boundary of Q in the positive direction.

Problem 3

Calculate

$$\oint_C x^2 e^y \, dx + y^2 e^x \, dy$$

where C is the same as in the previous problem.

Problem 4

Define Q as the square in $\mathbb{R}^2$ enclosing all the points (x, y) satisfying $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

Calculate

$$\oint_C \left(\frac{-y}{x^2 + y^2} \right) dx + \left(\frac{x}{x^2 + y^2} \right) dy$$

where C is the boundary of Q in the positive direction. You can use the fact that

$$\int_{-1}^{1} \frac{2}{x^2 + 1} \, dx = \pi.$$

Problem 5

Prof. Goofy applies the following argument to "show" that the integral in Problem 4 equals 0. But his argument is wrong. Point out the place where he makes a mistake.

Prof. Goofy's solution: Set $f_1 = \frac{-y}{x^2+y^2}$ and $f_2 = \frac{x}{x^2+y^2}$. Thus:

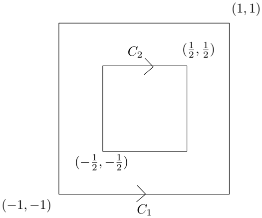
$$\frac{\partial f_1}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2}, \qquad \frac{\partial f_2}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Let D be the area enclosed by Q. By Green's theorem, we have:

$$\oint_C \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy = \iint_D \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} dxdy = 0$$

Problem 6

Suppose that C is the union of the two arcs C_1 and C_2 as shown in the following figure:



Calculate

$$\int_C (-y)\, dx + x\, dy.$$

Problem 7

Decide whether the following line integral is path independent. If so, calculate the integral on a piecewise smooth arc from point $(0, 0)$ to point $(1, 1)$ in 2d.

$$\int_C 2e^{x^2} (x \cos(2y)\, dx - \sin(2y)\, dy)$$

Problem 8

Decide whether the following line integral is path independent. If so, calculate the integral on a piecewise smooth arc from point $(0, 0, 0)$ to point $(1, 1, 1)$ in 3d.

$$\int_C (x^2 y \, dx - 4xy^2 \, dy + 8z^2 x \, dz)$$