1a) $f(x) = 1 + x - 3x^2$, at $c=1$

$f'(x) = 1 - 6x$, $f''(x) = -6$, $f^{(n)}(x) = 0$ $(n \ge 3)$

$\Rightarrow f(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \cdots$

$= -1 + (-5)(x-1) + \frac{(-6)}{2}(x-1)^2$

$= -1 - 5(x-1) - 3(x-1)^2$

1b) $f(x) = \sqrt{x}$. at $c=1$.

try to find the formula for $f^{(n)}(1)$.

$f^{(1)}(x) = \frac{1}{2}x^{-\frac{1}{2}}$, $f^{(2)}(x) = \frac{1}{2} \cdot (-\frac{1}{2}) x^{-\frac{1}{2}-1} = \frac{1}{2} \cdot (-\frac{1}{2}) x^{-\frac{3}{2}}$

$f^{(3)}(x) = \frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) x^{-\frac{5}{2}}$, so we can know the patterns for $f^{(n)}(x)$:

$f^{(n)}(x) = (-1)^{n-1} \cdot \frac{1}{2^n} \cdot (1 \cdot 3 \cdots (2n-3)) x^{-\frac{2n-1}{2}}$ $(n \ge 2)$

$\Rightarrow f^{(n)}(1) = \frac{(-1)^{n-1}}{2^n}(1 \cdot 3 \cdots (2n-3))$ $n \ge 2$.

$\Rightarrow f(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \cdots + \frac{(-1)^{n-1}(1 \cdot 3 \cdots (2n-3))}{2^n \cdot n!}(x-1)^n + \cdots$

2a) $f(x) = \frac{1}{1+2x}$ $c=0$.

Sol-1: $f'(x) = -\frac{1}{(1+2x)^2} \cdot 2$, $f''(x) = (-1)(-2)\frac{1}{(1+2x)^3} \cdot 2^2 \cdots$

$f^{(n)}(x) = (-1)^n \cdot n! \frac{2^n}{(1+2x)^{n+1}}$

$\Rightarrow f^{(n)}(0) = (-1)^n \cdot n! \cdot 2^n$

$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \cdots = 1 - 2x + 4x^2 + \cdots + (-1)^n 2^n \cdot x^n + \cdots = \sum_{n=0}^{+\infty} (-1)^n \cdot 2^n x^n$

Sol-2: Based on the geometric series:

$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$

then $\frac{1}{1+2x} = \frac{1}{1-(-2x)} \overset{t=-2x}{=} \frac{1}{1-t} = 1 + t + \cdots + t^n + \cdots = 1 - 2x + \cdots + (-2x)^n + \cdots$

2b) $g(x) = \frac{x}{1+2x} = x \cdot f(x)$

$$= x \cdot \sum_{n=0}^{+\infty} (-1)^n 2^n x^n$$

$$= \sum_{n=0}^{+\infty} (-1)^n \cdot 2^n x^{n+1}$$

$$= \sum_{n=1}^{+\infty} (-1)^{n-1} 2^{n-1} x^n$$

2c) $h(x) = \frac{1}{(1+2x)^2} = -\frac{f'(x)}{2}$

$f(x) = \sum_{n=0}^{+\infty} (-1)^n \cdot 2^n \cdot x^n$. differential both sides

$$f'(x) = \sum_{n=1}^{+\infty} (-1)^n 2^n \cdot n\, x^{n-1}$$

$$\Rightarrow h(x) = -\frac{f'(x)}{2} = \sum_{n=1}^{+\infty} (-1)^{n+1} \cdot 2^{n-1} n\, x^{n-1}$$

$$= \sum_{n=0}^{+\infty} (-1)^{n+2} 2^n (n+1) x^n = \sum_{n=0}^{+\infty} (-1)^n \cdot 2^n (n+1) x^n$$

2d) $k(x) = \frac{1}{(1-x)(2-x)}$ $\quad c=0$

$$k(x) = \frac{1}{1-x} - \frac{1}{2-x}$$

$$= (1+x+\cdots+x^n+\cdots) - \frac{1}{2} \cdot \frac{1}{1-\frac{x}{2}}$$

$$= (1+x+\cdots+x^n+\cdots) - \frac{1}{2}\left(1+\frac{x}{2}+\cdots+\left(\frac{x}{2}\right)^n+\cdots\right.$$

$$= \sum_{n=0}^{+\infty} \left(1-\frac{1}{2^{n+1}}\right) x^n$$

3. This problem provides a proof for "e" is not a rational number, and we need to apply the proof by contradiction here, means we first assume:

e is a rational number $\Leftrightarrow$ $e = \frac{p}{q}$ where $p, q \in N^+$ from the definition of the rational number.

$\uparrow$

Assumption.

Then we starts from the taylor expanding of e^x at $c=0$. for $(e^x)^{(n)} = e^x$, we have:

$$e^x = 1 + \frac{1}{1!}x + \cdots + \frac{1}{n!}x^n + E_n(x)$$

Due to the fact that e^x is a smooth function, so the order n can go to infinity which means we can choose the n arbitrary according to our need (this is important!)

And $E_n(x)$ is the error term, so $E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{e^\xi}{(n+1)!} x^{n+1}$ $\quad \xi \in (0, x)$

Now we ~~choose~~ set $x=1$, then:

$$\frac{p}{q} = e = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} + \frac{e^\xi}{(n+1)!} \quad \xi \in (0,1)$$

$$\Rightarrow n!\,\frac{p}{q} = n! + \frac{n!}{1!} + \cdots + \frac{n!}{n!} + \frac{e^\xi}{n+1} \quad \xi \in (0,1)$$

so first we want to make sure $LHS = n! \cdot \frac{r}{q}$ be an integer.

which need $n \geq q$, for $n! = 1 \cdot 2 \cdot 3 \cdots q \cdots n$ if $n \geq q$, q can be cancelled.

counter-example is $4! \cdot \frac{1}{5}$ is not an integer.

so our first condition for n : $\boxed{n \geq q}$ ①.

then for $RHS = n! + \frac{n!}{1!} + \cdots + \frac{n!}{n!} + \frac{e^{\xi}}{(n+1)}$.

All terms like $\frac{n!}{m!}$ $(n \geq m)$ is interger for $\frac{n!}{m!} = (m+1) \cdots n$ $(n \geq m)$

so inorder to get the contradiction, we just need to let $\frac{e^{\xi}}{(n+1)}$ is not an integer.

And: $0 < \frac{e^{\xi}}{n+1} < \frac{e^{1}}{(n+1)} < \frac{3}{n+1}$ $\quad (\xi \in (0,1), e < 3)$

so we just need $\frac{3}{n+1} < 1 \Leftrightarrow n > 2 \Leftrightarrow n \geq 3$. then $0 < \frac{e^{\xi}}{n+1} < 1$ which implies it's not an integer.

then $LHS = n! \cdot \frac{p}{q} \in N^{+} = RHS = n! + \cdots + \frac{n!}{n!} + \frac{e^{\xi}}{(n+1)} \notin N^{+}$ if we have $\begin{cases} n \geq q \\ n \geq 3 \end{cases} \Rightarrow n = \max\{q, 3\}$.

contradiction!

$\Rightarrow$ the assumption is wrong, the "e" is not a rational number.

Another question is about the "radius of convergence", like:

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k+1} \cdot x^k}{k} + E_n(x)$$

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} x^{n+1} = \frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}} \cdot x^{n+1} \quad \xi \in (0,x) \text{ or } \xi \in (x,0)$$

the "radius of convergence" just the set of x that make $E_n(x) \to 0$ when $n \to +\infty$.

consider $x = 1$. then :

$$\left|E_n(1)\right| = \left|\frac{(-1)^n}{n+1} \cdot \frac{1}{(1+\xi)^{n+1}}\right| < \frac{1}{n+1} \quad \text{for } \xi \in (0,1) \Rightarrow \frac{1}{1+\xi} < 1$$

so by sand-wich then $|E_n(1)| \to 0$ when $n \to +\infty$.

But if $x = a > 1$. we know that $\frac{a^n}{n+1} \to +\infty$ when $n \to +\infty$

And later you would learn how to compute the radius of convergence of a taylor series.

this example just shows the convergence interval for $\ln(1+x)$ is $(-1, 1]$