**MATH5070**
**Homework 3 solution**

1. (i) For any compact subinterval $t$ lies in, there exist an integer $M > 0$ such that $|t| < M$. Then
$$\left\|\frac{t^n A^n}{n!}\right\| \le \frac{\|A\|^n |t|^n}{n!} < \frac{\|A\|^n M^n}{n!},$$
where the series $\displaystyle\sum_{n=0}^{\infty} \frac{\|A\|^n M^n}{n!}$ converges by ratio test. Therefore $\displaystyle\sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$ converges uniformly by the Weierstrass M-test.

   (ii) We first differentiate the series term by term, then we get
$$\sum_{n=0}^{\infty} \frac{d}{dt}\left(\frac{t^n A^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{(n-1)!} = \left(\sum_{n=1}^{\infty} \frac{t^{n-1} A^{n-1}}{(n-1)!}\right) A = A\left(\sum_{n=1}^{\infty} \frac{t^{n-1} A^{n-1}}{(n-1)!}\right).$$
We have that $\displaystyle\sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{(n-1)!}$ converges uniformly on the given subinterval by the same reason as (i). Therefore we finally show that
$$\frac{d}{dt} \exp tA = (\exp tA)A = A(\exp tA).$$

   (iii) By (ii), we see that $\exp tA$ can be differentiated any number of times, thus it is smooth in $t$.

2. It suffices to show that $v_A$ is complete. For any $x_0 \in \mathbb{R}^n$, let $\gamma(t) = (\exp tA)(x_0)$. By Q1(iii), $\gamma(t)$ is smooth and $\gamma(0) = I_n x_0 = x_0$.
$$\frac{d}{dt}\gamma(t) = (\frac{d}{dt} \exp tA)x_0 = A(\exp At)x_0 = A\gamma(t) = v_A(\gamma(t)).$$
Therefore $\gamma(t)$ is the integral curve of $v_A$ through $x_0$, then $v_A$ is complete.

3. Choose any $t \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Let $x_1 = (\exp tA)x_0$. Consider $\gamma_1(s) = (\exp sA)x_1 = (\exp sA)(\exp tA)x_0$ and $\gamma_2(s) = (\exp(t+s)A)x_0$. We see that $\gamma_1(0) = x_1$ and $\gamma_2(0) = x_1$ and both $\gamma_1(s)$ and $\gamma_2(s)$ are integral curve of $v_A$. By the uniqueness, we have that $\gamma_1(s) = (\exp sA)(\exp tA)x_0 = (\exp(t+s)A)x_0 = \gamma_2(s)$ for any $t$ and $x_0$. Therefore, $(\exp sA)(\exp tA) = (\exp(t+s)A)$.

4. $\phi : \mathbb{R} \to GL(n)$ is a homomorphism, thus $\phi(0) = I_n$. Consider the derivative of $\phi$,
$$\phi'(t) = \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t) - \phi(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\phi(\Delta t) - \phi(0))\,\phi(t)}{\Delta t} = \phi'(0)\phi(t).$$
Choose an arbitrary $x_0 \in \mathbb{R}^n$. Define a curve $\gamma_0(t) : \mathbb{R} \to \mathbb{R}^n$ as $\gamma_0(t) = \phi(t)x_0$. We see that $\gamma_0(0) = x_0$ and
$$\gamma_0'(t) = \phi'(t)x_0 = \phi'(0)\phi(t)x_0 = \phi'(0)\gamma_0(t) = v_{\phi'(0)}(\gamma_0(t)),$$

therefore $\gamma_0(t)$ is an integral curve of $v_{\phi'(0)}$ passing through $x_0$ at 0. Meanwhile, $\gamma(t) = (\exp t(\phi'(0)))x_0$ is an integral curve of $v_{\phi'(0)}$ passing through $x_0$ at 0. By uniqueness, $\gamma_0(t) = \phi(t)x_0 = (\exp t(\phi'(0)))x_0 = \gamma(t)$ for any $x_0$. Therefore $\phi(t) = \exp tA$, where $A = \phi'(0)$.

5. (i)⇒(ii) Choose any $x_0 \in \mathbb{R}$ and $s \in \mathbb{R}$. Define two curves $\gamma_1(t) = (\exp tA)(\exp sB)x_0$ and $\gamma_2(t) = (\exp sB)(\exp tA)x_0$. We see that $\gamma_1(0) = \gamma_2(0) = (\exp sB)x_0$,

$$\gamma_1'(t) = A(\exp tA)(\exp sB)x_0 = A\gamma_1(t) = v_A(\gamma_1(t)),$$

$$\text{and } \gamma_2'(t) = (\exp sB)A(\exp tA)x_0 = A(\exp sB)(\exp tA)x_0 = A\gamma_2(t) = v_A(\gamma_2(t)).$$

So both $\gamma_1(t)$ and $\gamma_2(t)$ are integral curves of $v_A$ passing through $(\exp sB)x_0$ at 0. By uniqueness, $\gamma_1(t) = (\exp tA)(\exp sB)x_0 = (\exp sB)(\exp tA)x_0 = \gamma_2(t)$. Since $x_0$ and $s$ are arbitrary, we have $(\exp tA)(\exp sB) = (\exp sB)(\exp tA)$.
(ii)⇒(i) By assumption, we have $(\exp tA)(\exp tB) = (\exp tB)(\exp tA)$. Differentiate this equation twice, then we have

$$\begin{aligned} &A^2(\exp tA)(\exp tB) + 2A(\exp tA)(\exp tB)B + (\exp tA)(\exp tB)B^2 \\ &= (\exp tB)(\exp tA)A^2 + 2B(\exp tB)(\exp tA)A + B^2(\exp tB)(\exp tA). \end{aligned}$$

Taking $t = 0$ implies $AB = BA$.
(i)⟺(iii)

$$\begin{aligned} [v_A, v_B] &= \sum_i \left( \left( \sum_k (\sum_j a_{kj}x_j) \frac{\partial}{\partial x_k} \right) \left( \sum_l b_{il}x_l \right) - \left( \sum_k (\sum_j b_{kj}x_j) \frac{\partial}{\partial x_k} \right) \left( \sum_l a_{il}x_l \right) \right) \frac{\partial}{\partial x_i} \\ &= \sum_i \left( \left( \sum_k (\sum_j a_{kj}x_j) b_{ik} \right) - \left( \sum_k (\sum_j b_{kj}x_j) a_{ik} \right) \right) \frac{\partial}{\partial x_i} \\ &= \sum_i \left( \left( \sum_j \sum_k a_{kj}b_{ik}x_j \right) - \left( \sum_j \sum_k b_{kj}a_{ik}x_j \right) \right) \frac{\partial}{\partial x_i} \\ &= \sum_i \left( \sum_j ([BA]_{ij} - [AB]_{ij})\, x_j \right) \frac{\partial}{\partial x_i} \\ &= \sum_i ([(AB - BA)x]_i) \frac{\partial}{\partial x_i}. \end{aligned}$$

Hence $[v_A, v_B] = 0 \iff AB = BA$.

6. (i)⇒(ii) $x_0$ is an arbitrary point in $\mathbb{R}^n$.

$$\begin{aligned} \frac{d}{dt}\|(\exp tA)x_0\|^2 &= \frac{d}{dt}\langle(\exp tA)x_0, (\exp tA)x_0\rangle \\ &= \langle A(\exp tA)x_0, (\exp tA)x_0\rangle + \langle(\exp tA)x_0, A(\exp tA)x_0\rangle \\ &= \langle(A + A^T)(\exp tA)x_0, (\exp tA)x_0\rangle. \end{aligned}$$

$A^T = -A$ implies that $\|(\exp tA)x_0\|^2$ is a constant. Let $t = 0$, $\|(\exp tA)x_0\|^2 = \|x_0\|^2$. It follows $\|(\exp tA)x_0\|^2 = x_0^T(\exp tA)^T(\exp tA)x_0 = x_0^T x_0$ for any $x_0$. Therefore

$(\exp tA)^T(\exp tA) = I_n$, which means $\exp tA \in O(n)$ for all $t \in \mathbb{R}$.
(ii)⇒(i) $(\exp tA)^T(\exp tA) = I_n$ implies that $\|(\exp tA)x_0\|^2 = \|x_0\|^2$ for all $t \in \mathbb{R}$. Therefore $\frac{d}{dt}\|(\exp tA)x_0\|^2 = \langle (A + A^T)(\exp tA)x_0, (\exp tA)x_0\rangle = 0$. Since $x_0$ is an arbitrary point and $\exp tA$ is invertible, we have $A + A^T = 0$.

7. (i) It sufices to show that $[V, W]$ belongs to $\mathcal{V}$.

$$\begin{aligned}[V,W] &= (D_V(1) - D_W(x))\,\frac{\partial}{\partial x} + (D_V(0) - D_W(1))\,\frac{\partial}{\partial y} + (D_V(y) - D_W(xy+x))\,\frac{\partial}{\partial z} \\ &= -\frac{\partial}{\partial x} + 0 \cdot \frac{\partial}{\partial y} + (-y)\,\frac{\partial}{\partial z} = -W \in \mathcal{V}\end{aligned}$$

Hence $\mathcal{V}$ is involutuve.

(ii) $X$ and $Y$ are linear independent, and $X = W,\ Y = V - xW$, thus $X$ and $Y$ span $\mathcal{V}$. For any point $p \in \mathbb{R}^3$, we have

$$d\pi_p = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

So $d\pi_p(X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\partial}{\partial x}$ and $d\pi_p(Y) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial}{\partial y}$.

(iii) For any $p = (x_0, y_0, z_0) \in \mathbb{R}^3$,

$$\begin{aligned}\gamma_p^1 &= (x_0 + t, y_0, z_0 + y_0 t) \\ \gamma_p^2 &= (x_0, y_0 + t, z_0 + x_0 t)\end{aligned}$$

are the integral curves of $X$ and $Y$ respectively.

(iv) For any $p = (x_0, y_0, z_0) \in \mathbb{R}^3$, the integral manifold of $\mathcal{V}$ passing through $p$ can be represented as a parametric surface in $\mathbb{R}^3$:

$$(x, y, z) = G(s,t) = \gamma^2_{\gamma^1_p(t)}(s) = \gamma^2_{(x_0+t, y_0, z_0+y_0t)}(s) = (x_0+t, y_0+s, z_0+y_0t+x_0s+st).$$

The surface can also be written as $z = xy + z_0 - x_0y_0$. Therefore the integral manifolds of $\mathcal{V}$ are $S = \{(x,y,z) \in \mathbb{R}^3 \mid z - xy = c, c \in \mathbb{R}\}$.