MATH5070
Homework 2 solution

1. (a) Let $\gamma(t) = A + Bt$, then $\gamma(t)$ satisfies that $\gamma(0) = A$ and $\gamma'(0) = B$. It suffices to show that $df_A(B) = (f \circ \gamma)'(0) = A^tB + B^tA$.

$$df_A(B) = (f\circ\gamma)'(0) = \lim_{t\to 0} \frac{(A+Bt)^t(A+Bt) - A^tA}{t} = \lim_{t\to 0} B^tBt + A^tB + B^tA = A^tB + B^tA$$

(b) $f^{-1}(I_n) = \{A \mid A^tA = I_n\} = O(n)$. It suffices to show df_A is surjective. For any $C \in Sym_n$, consider $\frac{1}{2}AC$, then

$$df_A(\frac{1}{2}AC) = \frac{1}{2}A^tAC + \frac{1}{2}C^tA^tA = \frac{1}{2}C + \frac{1}{2}C^t = C.$$

(c) By regular value theorem, $O(n) = f^{-1}(I_n)$ has dimension $dim(\mathcal{M}_n) - dim(Sym_n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$.

(d) We claim that $A \in \mathcal{M}_n$ is invertible if and only if A is a regular point.
($\Rightarrow$) If A is invertible, then for any $C \in Sym_n$, we consider $\frac{1}{2}(A^t)^{-1}C$,

$$df_A(\frac{1}{2}(A^t)^{-1}C) = \frac{1}{2}A^t(A^t)^{-1}C + \frac{1}{2}C^tA^{-1}A = \frac{1}{2}C + \frac{1}{2}C^t = C.$$

Thus df_A is surjective.
($\Leftarrow$) If df_A is surjective, then there exists $B \in \mathcal{M}_n$ such that $A^tB + B^tA = I_n$. If A is not invertible, then there exists $x \neq 0 \in \mathbb{R}^n$ such that $Ax = 0$. Consider

$$0 < \|x\|^2 = x^tI_nx = x^t(A^tB + B^tA)x = (Ax)^tBx + (Bx)^tAx = 0,$$

which gives a contradiction!
Furthermore, $f(A)$ is invertible if and only if A is invertible, since $det(A^tA) = det(A)^2$.
Therefore,
$\{Regular\ points\} = \{A \mid A \in \mathcal{M}_n\ is\ invertible\}$
$\{Critical\ points\} = \{A \mid A \in \mathcal{M}_n\ is\ noninvertible\}$
$\{Critical\ value\} = \{A \mid A \in Sym_n \cap Im(f)\ is\ noninvertible\}$
$\{Regular\ value\} = Sym_n \setminus \{Critical\ value\}$

(e) It suffices to show that the set of non-invertible symmetric matrices in Sym_n is measure zero. $det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is basically a nonzero polynomial, and the set of non-invertible symmetric matrices is the set of roots of det, which is measure zero.

2. It suffices to show that both ι and $d\iota_p$ for all p are injective. If there exists p and q such that $\iota(p) = \iota(q)$, then

$$(\rho_1(p)\varphi_1(p), \ldots, \rho_r(p)\varphi_r(p), \rho_1(p), \ldots, \rho_r(p)) = (\rho_1(q)\varphi_1(q), \ldots, \rho_r(q)\varphi_r(q), \rho_1(q), \ldots, \rho_r(q)).$$

There exists some i such that $\rho_i(p) = \rho_i(q) \neq 0$. $\rho_i(p)\varphi_i(p) = \rho_i(q)\varphi_i(q)$ implies $\varphi_i(p) = \varphi_i(q)$ which contradicts that φ_i is a homeomorphism.
Given any $p \in M$, there exists i such that $\rho_i(p) \neq 0$. If there exists $v \neq 0 \in \mathbb{R}^n$ such that $d\iota_p(v) = 0$, then $d(\psi_i)_p(v) = 0$ and $d(\rho_i)_p(v) = 0$, where $d(\psi_i)_p(v) = \rho_i(p)d(\varphi_i)_p(v) + d(\rho_i)_p(v)\varphi_i(p)$. Since $\rho_i(p) \neq 0$, we have $d(\varphi_i)_p(v) = 0$ which contradicts φ_i is a homeomorphism.

3. (Optional) There exists $F : J \to I$ such that $\tilde{\gamma} = \gamma \circ F$ means

$$\begin{array}{lc} \tilde{\gamma}'(t) = w_{\tilde{\gamma}(t)} = f(\tilde{\gamma}(t)) v_{\tilde{\gamma}(t)} = & f(\gamma(F(t))) v_{\gamma(F(t))} \\ & \| \\ (\gamma(F(t)))' = \gamma'(F(t)) F'(t) = & F'(t) v_{\gamma(F(t))} \end{array}$$

ODEs: $F'(t) = f(\gamma(F(t)))$ and $F(0) = 0$ has a unique solution, which shows the existence and uniqueness of $F(t)$. Since f is non-vanishing, $F'(t) = f(\gamma(F(t)))$ implies $F(t)$ is monotone, thus there exists an inverse of $F(t)$. Therefore $F : I \to J$ must be a diffeomorphism.