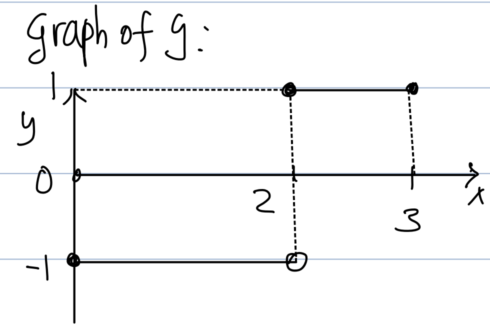
HW6

7.3 Q13 $g(x) = \begin{cases} -1 & 0 \leq x < 2 \\ 1 & 2 \leq x \leq 3 \end{cases}$

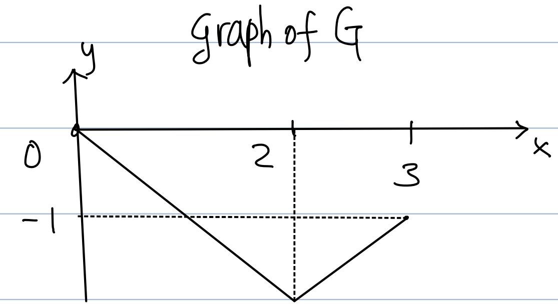



For $0 \leq x < 2$, $G(x) = \int_0^x g = \int_0^x -1 = -x$

For $2 \leq x \leq 3$ $G(x) = \int_0^x g = \int_0^2 g + \int_2^x g = G(2) + \int_2^x g$

$= -2 + \int_2^x 1 = -2 + x - 2 = x - 4$

$\therefore G(x) = \begin{cases} -x, & 0 \leq x < 2 \\ x-4, & 2 \leq x \leq 3 \end{cases}$




No! $G'(x)$ doesn't exist at $x = 2$.

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7.3 Q15 Since $f$ is continuous on $\mathbb{R}$, Fundamental Thm of Calculus implies $h(x) = \int_0^x f(t)\,dt$ is differentiable on $\mathbb{R}$ and $h'(x) = f(x)$, $\forall x \in \mathbb{R}$.

By Chain rule, $h(x+c)$ and $h(x-c)$ are also differentiable on $\mathbb{R}$. Hence

$g(x) = \int_{x-c}^{x+c} f = \int_0^{x+c} f - \int_0^{x-c} f = h(x+c) - h(x-c)$ is differentiable on $\mathbb{R}$, and

$$g'(x) = h'(x+c) - h'(x-c) = f(x+c) - f(x-c)$$

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7.3 Q16

Since $f$ is continuous on $[0,1]$, Fundamental Thm of Calculus implies $\int_0^x f$ and $\int_x^1 f = -\int_1^x f$ are differentiable on $[0,1]$ and

$$\frac{d}{dx}\int_0^x f = f(x) \quad \text{and} \quad \frac{d}{dx}\int_x^1 f = -f(x).$$

Since $\int_0^x f = \int_x^1 f$, $\forall x \in [0,1]$, we have

$$f(x) = -f(x), \quad \forall x \in [0,1]$$

$$\Rightarrow \quad f(x) = 0, \quad \forall x \in [0,1].$$

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7.3 Q22

$$h(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \ (m,n \geq 1, \text{relatively prime}) \\ 1, & \text{if } x = 0 \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

Then $$\operatorname{sgn}(h)(x) = \begin{cases} 1, & \text{if } x = \frac{m}{n} \ (m,n \geq 1, \text{relatively prime}) \\ 1, & \text{if } x = 0 \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

= Dirichlet function

which is not Riemann integrable.

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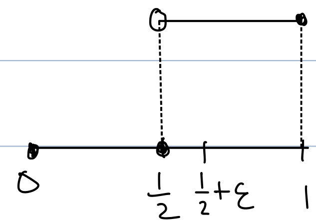
7.4 Q7

(a) $\forall \varepsilon > 0$, let

$$P_\varepsilon = \left(0, \tfrac{1}{2}, \tfrac{1}{2}+\varepsilon, 1\right)$$




Then $U(g, P_\varepsilon) = 0 \cdot \frac{1}{2} + 1 \cdot \left(\frac{1}{2}+\varepsilon - \frac{1}{2}\right) + 1 \cdot \left(1 - \left(\frac{1}{2}+\varepsilon\right)\right)$

$$= \frac{1}{2}$$

and $L(g, P_\varepsilon) = 0 \cdot \frac{1}{2} + 0 \cdot \left(\frac{1}{2}+\varepsilon - \frac{1}{2}\right) + 1 \cdot \left(1 - \left(\frac{1}{2}+\varepsilon\right)\right)$

$$= \frac{1}{2} - \varepsilon$$

Hence $\frac{1}{2} - \varepsilon = L(g, P_\varepsilon) \leq L(g) \leq U(g) \leq U(g, P_\varepsilon) = \frac{1}{2}$.

Since $\varepsilon > 0$ is arbitrary, we have

$$L(g) = U(g) = \frac{1}{2}$$

(b) Since $g$ is integrable (both Riemann & Darboux), changing value at one point is still integrable and with the same integral $\int_0^3 g = L(g) = U(g) = \frac{1}{2}$

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7.4 Q9. Since for any subinterval $[x_{i-1}, x_i]$ on $[a,b]$

$$\inf_{[x_{i-1},x_i]} f_1(x) \le f_1(x) \qquad (\forall x \in [x_{i-1}, x_i])$$

and
$$\inf_{[x_{i-1},x_i]} f_2(x) \le f_2(x)$$

$$\Rightarrow \inf_{[x_{i-1},x_i]} f_1(x) + \inf_{[x_{i-1},x_i]} f_2(x) \le f_1(x) + f_2(x), \quad \forall x \in [x_{i-1}, x_i]$$

$$\Rightarrow \inf_{[x_{i-1},x_i]} f_1(x) + \inf_{[x_{i-1},x_i]} f_2(x) \le \inf_{[x_{i-1},x_i]} \left[f_1(x) + f_2(x)\right]$$

Therefore, $\forall$ partition $P$ of $[a,b]$,

$$L(f_1; P) = \sum_{i=1}^{n} \left( \inf_{[x_{i-1},x_i]} f_1(x) \right) \cdot (x_i - x_{i-1})$$

$$L(f_2; P) = \sum_{i=1}^{n} \left( \inf_{[x_{i-1},x_i]} f_2(x) \right) \cdot (x_i - x_{i-1})$$

$$\Rightarrow L(f_1; P) + L(f_2, P) \le \sum_{i=1}^{n} \left( \inf_{[x_{i-1},x_i]} (f_1(x) + f_2(x)) \right) (x_i - x_{i-1})$$

$$= L(f_1 + f_2\ P)$$

$$\le L(f_1 + f_2) \quad \text{———————} \quad (*)$$

Now, $\forall \varepsilon > 0$, $\exists P_1$ s.t. $L(f_1) < L(f_1, P_1) + \varepsilon$

and $\exists P_2$ s.t. $L(f_2) < L(f_2, P_2) + \varepsilon$.

Let $P = P_1 \cup P_2$ be the common refinement of $P_1$ & $P_2$.

Then $L(f_1) < L(f_1, P) + \varepsilon$ since $L(f,P) \geq L(f,P_1)$

$L(f_2) < L(f_2, P) + \varepsilon$ $L(f,P) \geq L(f,P_2)$

$$\therefore \quad L(f_1) + L(f_2) < L(f_1, P) + L(f_2, P) + 2\varepsilon$$

$$\leq L(f_1 + f_2) + 2\varepsilon \quad (\text{by } (*))$$

Since $\varepsilon > 0$ is arbitrary, we have

$$L(f_1) + L(f_2) \leq L(f_1 + f_2). \quad \#$$

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7.4 Q10

Dirichlet function $D(x) = \begin{cases} 1, & \text{if } x \text{ rational} \\ 0, & \text{if } x \text{ irrational} \end{cases}$ on $[0,1]$

Then $L(D; P) = 0$ $\forall$ partition $P$

$\Rightarrow$ $L(D) = 0$

Consider $f(x) = \begin{cases} 0, & \text{if } x \text{ rational} \\ 1, & \text{if } x \text{ irrational} \end{cases}$ on $[0,1]$

Then similarly $L(f; P) = 0$, $\forall$ partition $P$

$\Rightarrow$ $L(f) = 0$.

$\therefore$ $L(D) + L(f) = 0$.

However $(D+f)(x) = 1$ $\forall x \in [0,1]$ is integrable & $\int_0^1 (D+f) = 1$

$\Rightarrow$ $L(D+f) = 1 > 0 = L(D) + L(f)$ $\#$