Remark: As mentioned before, example 47 is equivalent to

find f such that $M dx + N dy + L dz = df$.

In such a case (the "diff. form") $M dx + N dy + L dz$ is called exact.

(i.e. a "form" is exact $\Leftrightarrow$ it is a "total differential")

Remark: To prove Thm 10 in $\mathbb{R}^2$, we need the Green's Thm

(in $\mathbb{R}^3$, we need the Stokes' Thm)

Thm 11 (Green's Theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω;

C be a piecewise "smooth" simple closed anti-clockwise oriented curve enclosing a region R which lies entirely in Ω.

Then

- Normal Form

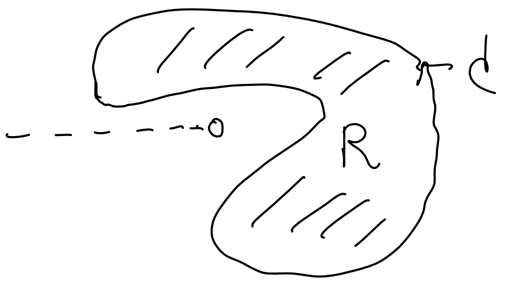
$$\oint_C \vec{F} \cdot \hat{n}\, ds = \boxed{\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx dy}$$

- Tangential Form

$$\oint_C \vec{F} \cdot \hat{T}\, ds = \boxed{\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy}$$

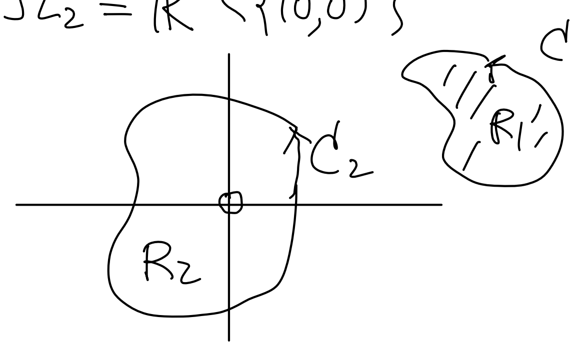
(Remark: The two forms are equivalent)

Note: $\Omega_1 = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Thm applies, since $R \subset \Omega_1$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



Green's Thm applies, since $R_1 \subset \Omega_2$

$(0,0) \in R_2$, but $(0,0) \notin \Omega_2$

$\Rightarrow R_2 \not\subset \Omega_2$,

Green's Thm doesn't apply.

eg 48 Verify both forms of Green's Thm for

$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j}$ on $\Omega = \mathbb{R}^2$, is C^∞.

C = unit circle $= \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$

Then R = region enclosed by $C = \{x^2+y^2<1\}$ the unit disc.

(We also write $C = \partial R$ boundary of R)

Soln: $M = x-y$ & $N = x$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1 \quad ; \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0$$

On curve C: $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

(with correct orientation: anti-clockwise)

Normal Form

$$\text{L.H.S.} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (\cos t - \sin t)\, d\sin t - \cos t\, d(\cos t)$$

$$= \int_0^{2\pi} \cos^2 t\, dt = \pi \qquad \text{(check!)}$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\, dx\,dy = \pi$$

<u>Tangential Form</u>

$$\text{L.H.S} = \oint_C M\,dx + N\,dy$$

$$= \int_0^{2\pi} (\cos t - \sin t)\, d\cos t + \cos t\, d\sin t$$

$$= \cdots = 2\pi \qquad \text{(check!)}$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1-(-1))\, dx\,dy = 2\pi$$

(<u>Note</u>: This example shows that even the 2 forms are equivalent, the <u>values</u> involved may differ.)

Pf of Green's Thm (tangential form)

Recall: A region R is of special type:

type (1): If $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

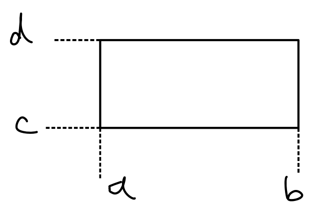
for some continuous functions $g_1(x)$ & $g_2(x)$.

type (2): If $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

for some continuous functions $h_1(x)$ & $h_2(x)$.

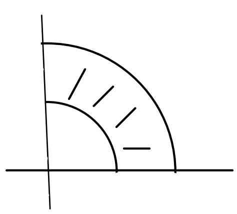
Now: If R is both type (1) and type (2), it said to be simple.

eg 48: (i)

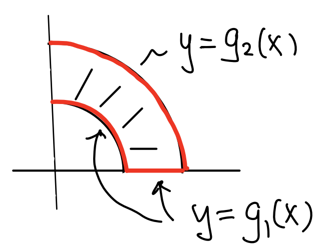


rectangle is simple

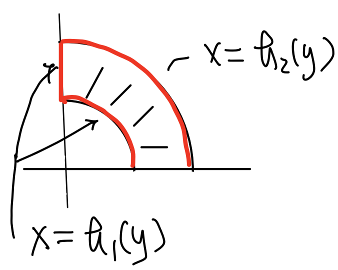
(ii)



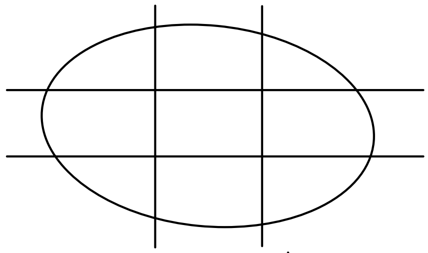
simple — type (1)



simple — type (2)



(iii')



2 intersections at most

2 intersections at most

$\Rightarrow$ simple

$$\forall a \in \mathbb{R}: \quad \left.\begin{array}{l} \#\{\partial R \cap \{x=a\}\} \leq 2 \\ \#\{\partial R \cap \{y=a\}\} \leq 2 \end{array}\right\} \Rightarrow \text{simple}$$

(provided ∂R is piecewise smooth)

(Proof : omitted)