(Cont'd)

"(c) ⇒ (a)" (it requires us to solve a system of PDE.)

Assume $n = 2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ be conservative.

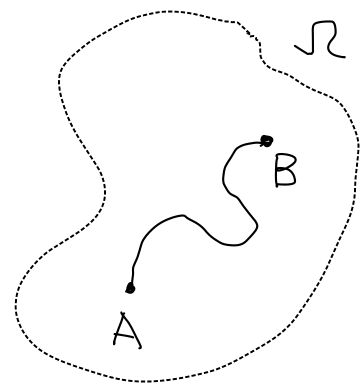
Then we need find $f \in C^1$ s.t.

$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \vec{\nabla} f = \vec{F} = M\hat{i} + N\hat{j}$$

partial differential equation

i.e. finding $f \in C^1$ s.t. $\begin{cases} \frac{\partial f}{\partial x} = M \\ \frac{\partial f}{\partial y} = N \end{cases}$ (a system of PDE)

(equivalent to $df = M\,dx + N\,dy$)



Fix a point $A \in \Omega$

Then for any point $B \in \Omega$,

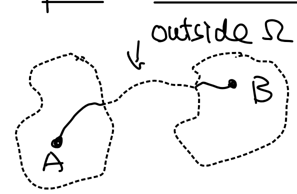
define

$$f(B) = \int_A^B \vec{F} \cdot \hat{T}\,ds = \underline{\text{common value}} \text{ of } \int_C \vec{F} \cdot \hat{T}\,ds \text{ for any } C \text{ from } A \text{ to } B.$$

Since $\vec{F}$ is conservative, $f(B)$ is well-defined.

We've also used the assumption that Ω is <u>(path) connected</u>.

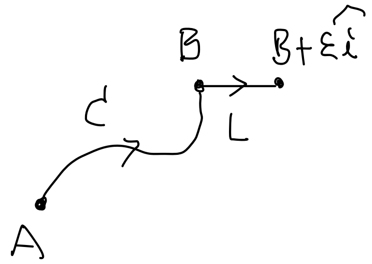
Otherwise, there is no path from A to B



if A, B belong to different connected components.

Claim: $\vec{F} = \vec{\nabla} f$

Pf of Claim
$$\frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{f(B+\varepsilon \hat{i}) - f(B)}{\varepsilon}$$



Let L be the horizontal straight line segment from B to $B+\varepsilon\hat{i}$ with $|\varepsilon|$ sufficiently small such that $B+\varepsilon\hat{i} \in \Omega$ (possible since Ω is open)

Then
$$f(B+\varepsilon\hat{i}) = \int_A^{B+\varepsilon\hat{i}} \vec{F}\cdot\hat{T}\,ds = \int_{C+L} \vec{F}\cdot\hat{T}\,ds$$
$$= \int_C \vec{F}\cdot\hat{T}\,ds + \int_L \vec{F}\cdot\hat{T}\,ds$$
$$= f(B) + \int_L \vec{F}\cdot\hat{T}\,ds$$

$$\frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon} = \frac{1}{\varepsilon}\int_L \vec{F}\cdot\hat{T}\,ds = \frac{1}{\varepsilon}\int_L \vec{F}\cdot d\vec{r}$$

Since L can be parametrized by $(x+t, y)$, $0 \le t \le \varepsilon$ if $B = (x,y)$, we have

$$\frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon} = \frac{1}{\varepsilon}\int_0^{\varepsilon} (M\hat{i}+N\hat{j})\cdot\left((x+t)'\,\hat{i} + y'\,\hat{j}\right)dt$$

$$= \frac{1}{\varepsilon} \int_0^{\varepsilon} M(x+t, y)\, dt$$

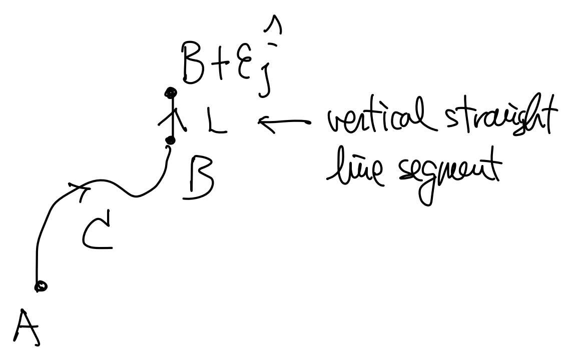
$$\longrightarrow M(x,y) \quad \text{as } \varepsilon \to 0 \qquad (\text{Ex!})$$

(by Mean Value Thm & M is cts ($\vec{F}$ is cts)
or Fundamental Thm of Calculus)

$$\therefore \quad \frac{\partial f}{\partial x}(B) = \frac{\partial f}{\partial x}(x,y) = M(x,y)$$

Similarly $\quad \frac{\partial f}{\partial y}(B) = N(x,y)$

by considering



So $\vec{\nabla} f = \vec{F}$

Since $\vec{F}$ is cts, $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$ are cts (f is clearly cts)

$\therefore$ f is C^1 ※

Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

"n=3" If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ (on $\Omega \subset \mathbb{R}^3$) connected open

then
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\[2mm] \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y} \\[2mm] \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$

"n=2" If $\vec{F} = M\hat{i} + N\hat{j}$ (on $\Omega \subset \mathbb{R}^2$) connected open

then $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Pf: $\vec{F}$ conservative $\overset{\text{Thm 9}}{\Longrightarrow}$ $\vec{F} = \vec{\nabla} f$ for some function f

i.e. $\dfrac{\partial f}{\partial x}\hat{i} + \dfrac{\partial f}{\partial y}\hat{j} + \dfrac{\partial f}{\partial z}\hat{k} = M\hat{i} + N\hat{j} + L\hat{k}$

$\vec{F} \in C^1 \Rightarrow f \in C^2$

Hence mixed derivatives thm (Clairaut's Thm)

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial N}{\partial x} \\[2mm] \dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial L}{\partial y} \\[2mm] \dfrac{\partial L}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial M}{\partial z} \end{cases}$$

("n=2" is included) $\#$

eg 42: Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is not conservative in $\mathbb{R}^2$.

Soln: $(\vec{F} \in C^\infty)$ $\begin{cases} M = 1 \\ N = x \end{cases} \Rightarrow \begin{cases} \frac{\partial M}{\partial y} = 0 \\ \frac{\partial N}{\partial x} = 1 \end{cases}$ $\neq$

$\Rightarrow \vec{F}$ is not conservative. ※

Remark (Important)

For a C^1 vector field $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$\boxed{\vec{F} \text{ conservative}} \underset{?}{\overset{\text{Cor. to Thm 9}}{\rightleftarrows}} \boxed{\begin{array}{c} M, N, L \text{ satisfy the system} \\ \text{of PDEs in the Cor to Thm 9} \end{array}}$$

Answer: NOT TRUE in general, needs extra condition on the domain Ω ("connected" is not enough)

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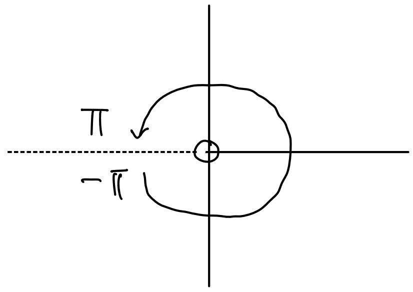
eg43 Consider the vector field

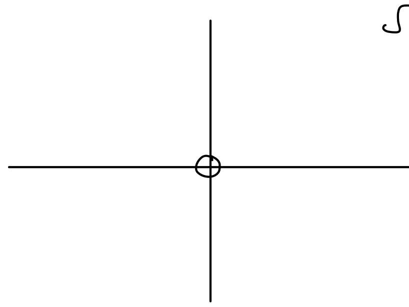
$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

and the domains $\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$

Ω_1 doesn't include the negative x-axis (and origin)

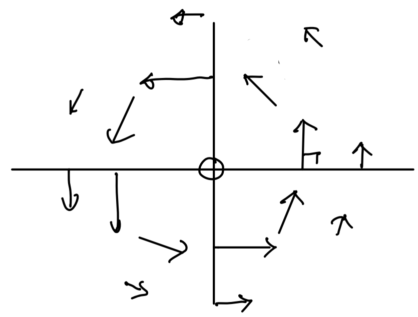




Ω_2 doesn't include the origin (but include the negative x-axis)

In polar coordinates

$$\vec{F} = -\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}$$



$\Rightarrow$
- $\vec{F}$ rotates around the origin anti-clockwisely
- $\|\vec{F}\| = \frac{1}{r} \to 0$ as $r \to +\infty$
- $\|\vec{F}\| = \frac{1}{r} \to \infty$ as $r \to 0$ $\Rightarrow$ $\vec{F}$ cannot be extended to a C^1 vecta field on $\mathbb{R}^2$.

Besides $(0,0)$, $\vec{F}$ is C^1. Hence

$\vec{F}$ is C^1 on Ω_1, and also

$\vec{F}$ is C^1 on Ω_2.

Questions : Is $\vec{F}$ conservative on Ω_1 ?

Is $\vec{F}$ conservative on Ω_2 ?

Soln: (1) For Ω_1, and $(x,y)\in\Omega_1$ can be expressed in polar coordinates by

$$\begin{cases} r>0 \\ -\pi<\theta<\pi \end{cases} \qquad (x=r\cos\theta,\ y=r\sin\theta)$$

Straight inequalities $\Rightarrow$ doesn't include negative x-axis

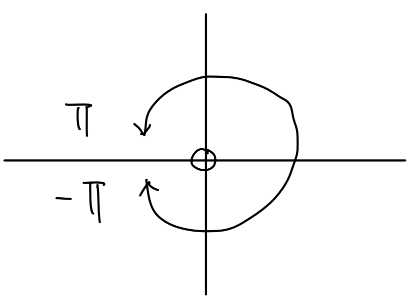
Define $f(x,y)=\theta$ "smooth" on Ω_1 (Check!)

Then
$$\begin{cases} \dfrac{\partial f}{\partial x}=\dfrac{\partial\theta}{\partial x}=-\dfrac{\sin\theta}{r} \\[2ex] \dfrac{\partial f}{\partial y}=\dfrac{\partial\theta}{\partial y}=\dfrac{\cos\theta}{r} \end{cases} \qquad \text{(check!)}$$

$$\Rightarrow \vec{F}=-\frac{\sin\theta}{r}\hat{i}+\frac{\cos\theta}{r}\hat{j}=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}=\vec{\nabla}f$$

$\Rightarrow$ $\vec{F}$ is conservative on Ω_1

(2) For Ω_2, the "function" $f(x,y)=\theta$ <u>cannot be extended</u> to a "smooth" function on (the whole) Ω_2.



the "function" $f=\theta$ "jumps" at the negative x-axis

$\Rightarrow$ f cannot be extended to a <u>continuous</u> function across the negative x-axis.

To show that $\vec{F}$ is not conservative on Ω_2, we consider a closed curve $C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [-\pi, \pi]$

(unit circle in Ω_2, but it is not a curve in Ω_1)

Then $\oint_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} \left(-\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j} \right) \cdot \left((\cos t)'\hat{i} + (\sin t)'\hat{j} \right) dt$

$\uparrow r=1$

$= \int_{-\pi}^{\pi} \left((-\sin\theta)^2 + \cos^2\theta \right) dt = \int_{-\pi}^{\pi} dt$

$= 2\pi \neq 0$

By Thm 9, $\vec{F}$ is not conservative. #