Conservative Vector Field

Def 14: Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be <u>conservative</u> if

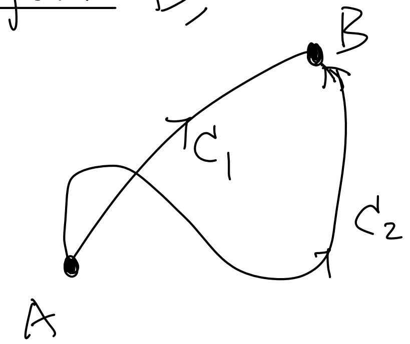
$$\int_C \vec{F} \cdot \hat{T}\, ds \left(= \int_C \vec{F} \cdot d\vec{r} \right)$$ along an <u>oriented</u> curve C in Ω <u>depends only</u> on the <u>starting</u> point and <u>end</u> point of C.

<u>Note</u>: This is usually referred as "<u>path independent</u>".

i.e. If C_1 & C_2 are <u>oriented</u> curves with the same <u>starting point</u> A and <u>end point</u> B,

then

$$\int_{C_1} \vec{F} \cdot \hat{T}\, ds = \int_{C_2} \vec{F} \cdot \hat{T}\, ds$$



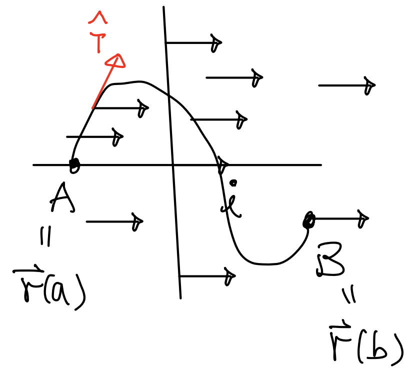
(so the value only depends on the points A & B (& direction))

<u>Notation</u>: If $\vec{F}$ is <u>conservative</u>, we sometimes write

$$\boxed{\int_A^B \vec{F} \cdot \hat{T}\, ds}$$ to denote the <u>common value</u> of $\int_C \vec{F} \cdot \hat{T}\, ds$ along <u>any</u> oriented curve C <u>from A to B</u>.

eg 41: $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$

$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, \quad a \le t \le b$



Then $\displaystyle\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$

$$= \int_a^b \hat{i}\cdot\left(x'(t)\hat{i} + y'(t)\hat{j}\right)dt$$

$$= \int_a^b x'(t)\,dt$$

$$= \underset{\uparrow}{x(b)} - \underset{\uparrow}{x(a)}$$

x-coordinates at $\vec{r}(b)$ & $\vec{r}(a)$ respectively

$\therefore \int_C \vec{F}\cdot\hat{T}\,ds$ depends only on the starting point $\vec{r}(a)$ & end point $\vec{r}(b)$ $\Rightarrow$ $\vec{F}$ is conservative. $\#$

(Note: $\vec{F} = \vec{\nabla} f$ where $f(x,y) = x$)

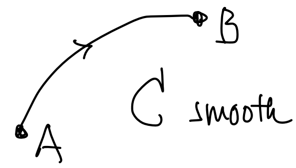
<u>Thm 8</u> (<u>Fundamental Theorem of Line Integral</u>)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, and $\vec{F} = \vec{\nabla} f$ be the gradient vector field of f. Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = f(B) - f(A)}$$

Pf: Part 1 Assume C is a smooth curve parametrized by

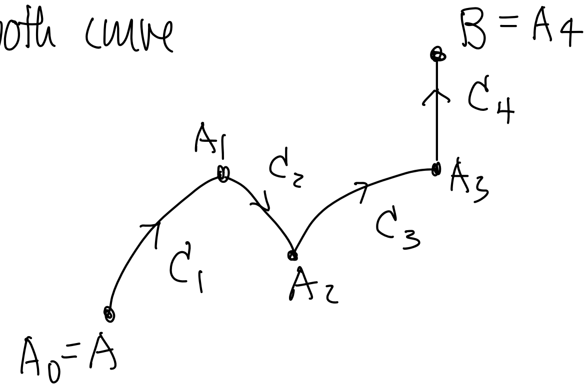
$$\vec{r}(t), \quad a \leq t \leq b$$



Then
$$\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$
$$= \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$
$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt \quad \text{(Chain rule)}$$
$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{(Fundamental Thm of Calculus)}$$
$$= f(B) - f(A) \quad \divideontimes$$

Part 2 For a general piecewise smooth curve

$$C = C_1 \cup C_2 \cup \cdots \cup C_k$$



($= C_1 + C_2 + \cdots + C_k$ in order to indicate that they are joining end-by-end and the orientation C_i, $i=1,\cdots,k$ are correct wrt the orientation of C)

where C_i is smooth going from A_{i-1} to A_i.

(then $A_0 = A$, $A_k = B$)

Then part 1 implies

$$\int_C \vec{F} \cdot \hat{T}\, ds = \int_{\sum_{i=1}^{k} C_i} \vec{F} \cdot \hat{T}\, ds$$

$$= \sum_{i=1}^{k} \int_{C_i} \vec{F} \cdot \hat{T}\, ds$$

$$= \sum_{i=1}^{k} \left[f(A_i) - f(A_{i-1}) \right] \qquad \text{(by part 1)}$$

$$= f(A_k) - f(A_0)$$

$$= f(B) - f(A) \qquad \#$$

Is the converse of Thm 8 correct?

Yes (under a further condition on the domain Ω)

Thm 9 Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open and (path) connected. $\vec{F}$ is a continuous vector field on Ω. Then the following are equivalent.

(a) $\exists$ a C^1 function $f: \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f$$

(b) $\oint_C \vec{F} \cdot d\vec{r} = 0$ along any closed curve C on Ω.

(c) $\vec{F}$ is conservative.

<u>Remarks</u>: (1) The function f in (a) of Thm 9 is called the <u>potential function</u> of $\vec{F}$. It is <u>unique up to</u> an <u>additive constant</u>:

$$\vec{\nabla}(f+c)=\vec{F}, \qquad \forall \text{ const. } c.$$

(2) $\vec{F}=M\hat{i}+N\hat{j}+L\hat{k}=\vec{\nabla}f \Leftrightarrow Mdx+Ndy+Ldz=df$

(Same for 2-dim)

In this case, $Mdx+Ndy+Ldz$ (or $Mdx+Ndy$ in dim. 2) is called an <u>exact differential form</u>.

<u>Pf</u>: "(a) $\Rightarrow$ (b)"

If f is C^1 and $\vec{F}=\vec{\nabla}f$ and

$\vec{r}:[a,b]\to\Omega$ parametrizes the closed curve C

then $\vec{r}(a)=\vec{r}(b) \overset{\text{denote}}{=\!=} A$

Fundamental Thm of Line Integral $\Rightarrow$

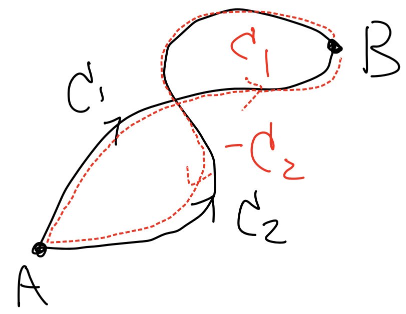
$$\oint_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b))-f(\vec{r}(a)) = f(A)-f(A)=0$$

"(b) $\Rightarrow$ (c)" Suppose C_1 & C_2 are oriented curves with starting point A and end point B.

Then $C_1 - C_2$ (i.e. $C_1 \cup (-C_2)$)

is an oriented closed curve

(with starting point = A = end point)



Then by part (b)

$$0 = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{(-C_2)} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

Since C_1 & C_2 are arbitrary, $\vec{F}$ is conservative.

"(c) ⇒ (a)" (it requires us to solve a system of PDE.)

(to be cont'd)