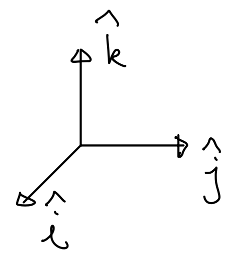
Vector Analysis

Notation : Usually in textbooks, vectors are denoted by boldface **i**, but hard to do it on screen, so my notation of vectors are:

$$\begin{cases} \text{general vectors : } \vec{V}, \vec{F}, \vec{r}, \vec{\nabla}, \cdots & \text{(differential operator)} \\ \text{unit vectors : } \hat{i}, \hat{j}, \hat{k}, \hat{n}, \hat{T}, \cdots \end{cases}$$



and use $\|\vec{V}\|$ to denote the length of $\vec{V}$ to avoid confusion with absolute value $|x|$

Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)
(path integrals)

Def 9 : The line integral of a function f on a curve (path, line) C with parametrization

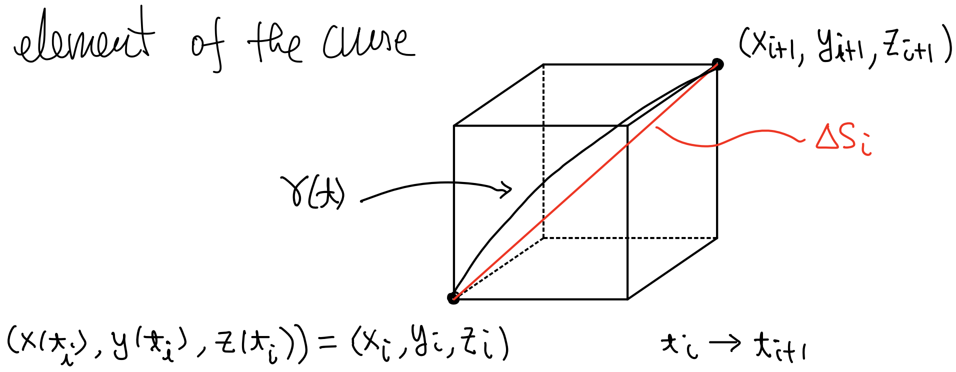
$$\begin{array}{rccc} \vec{r} : & [a,b] & \longrightarrow & \mathbb{R}^3 \\ & \psi & & \psi \\ \text{(position vector)} & t & \longmapsto & (x(t), y(t), z(t)) \end{array}$$

is
$$\int_C f(\vec{r})\, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\vec{r}(t_i)) \Delta s_i$$

where P is a partition of $[a,b]$, and

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

i.e. ds = length element of the curve



Remarks:

(1) If $f \equiv 1$,
$$\boxed{\int_C ds = \text{arc-length of } C}$$

(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $\vec{r}(t)$.

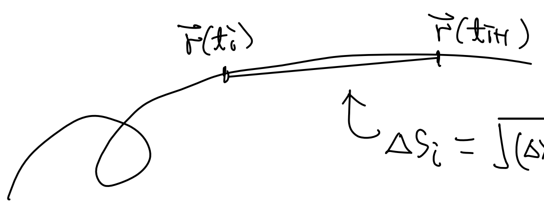
Def 9' (Formula for line integral)

Notations as in Def 9, then

$$\int_C f(\vec{r})\,ds = \int_a^b f(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$$

where $\quad \vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

Since



$$\Delta S_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$$= \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

$$\simeq \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t_i$$

$$= \|\vec{r}\,'(t_i)\|\ \Delta t_i$$

Remarks (1) "$ds = \|\vec{r}'(t)\|dt$" is usually referred as the arc-length element,

where $\vec{r}'(t) = (x'(t), y'(t), z'(t))$ and $\|\vec{r}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

(2) Suppose the curve C is parametrized by a new parameter $\tilde{t}$

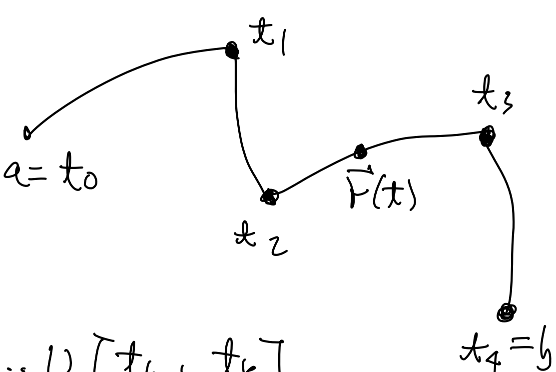
$$\begin{array}{ccc} t & \longleftrightarrow & \tilde{t} \\ \cap & & \cap \\ [a,b] & & [\tilde{a}, \tilde{b}] \end{array} \qquad \left(t \Leftrightarrow \tilde{t} \text{ is increasing } \frac{d\tilde{t}}{dt} > 0, \ \frac{dt}{d\tilde{t}} > 0 \right)$$

then $ds = \|\vec{r}'(t)\|dt = \left\|\frac{d\vec{r}}{dt}(t)\right\| dt$

$$= \left\|\frac{d\vec{r}}{d\tilde{t}} \cdot \frac{d\tilde{t}}{dt}\right\| dt = \left\|\frac{d\vec{r}}{d\tilde{t}}\right\| \left|\frac{d\tilde{t}}{dt}\right| dt = \left\|\frac{d\vec{r}}{d\tilde{t}}\right\| d\tilde{t}$$

$\therefore$ ds and hence $\int_C f(\vec{r})\,ds$ is <u>independent</u> of the parametrization of C.

(3) If $\vec{r}(t)$ is only piecewise differentiable, then the RHS of <u>Def 9'</u> becomes a sum:



If $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{i-1}, t_i] \cup \cdots \cup [t_{k-1}, t_k]$

$t_0 = a$, $t_k = b$; t_{i-1}, t_i: non-differentiable point for $\vec{r}(t)$

such that $\vec{r}\big|_{[t_{i-1}, t_i]}$ is differentiable, then

$$\boxed{\int_C f(\vec{r})\,ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\vec{r}(t))\|\vec{r}'(t)\|\,dt}$$

<u>eg32</u>: $f(x,y,z) = x - 3y^2 + z$

C = (straight) line segment joining the origin and $(1,1,1)$

Find $\int_C f(x,y,z)\,ds$

<u>Soln</u>:

Parametrize C by

$$\vec{r}(t) = (0,0,0) + t\,[(1,1,1) - (0,0,0)]$$

$$= (t, t, t) \qquad 0 \leq t \leq 1$$

(i.e $x(t) = t$, $y(t) = t$, $z(t) = t$)

$$\Rightarrow \quad \vec{r}'(t) = (1,1,1), \quad \forall\, t \in [0,1]$$

$$\Rightarrow \quad \int_C f(\vec{r})\,ds = \int_0^1 f(t,t,t)\,\|\vec{r}'(t)\|\,dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt$$

$$= 0 \qquad \text{(Check!)}$$

#

eg33 : Let C be a curve in $\mathbb{R}^2$ (plane curve) (i.e. $z(t) \equiv 0$)
and it has 2 parametrizations

$$\vec{r}_1(t) = (\cos t, \sin t), \quad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\vec{r}_2(t) = (\sqrt{1-t^2}, -t), \quad t \in [-1, 1]$$

Suppose $f(x,y) = x$. Find $\int_C f(x,y)\,ds$.

(We simply omit the z-variable, as C is a plane curve and f is indep. of z)

Soln: (1) $\vec{r}_1(t) = (\cos t, \sin t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\int_C f\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \, \| (\cos t, \sin t)' \| \, dt$$

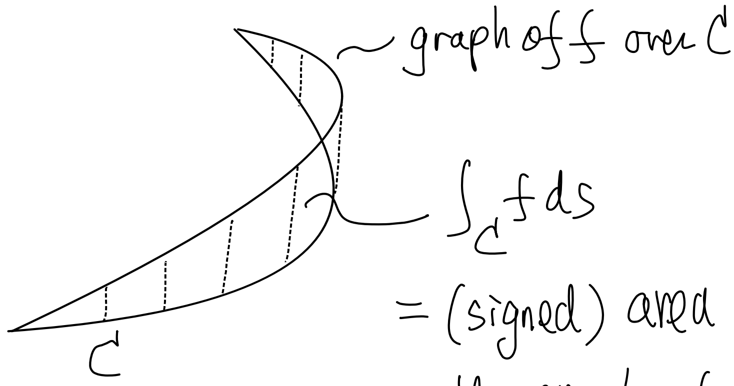
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \, dt = 2 \qquad \text{(check!)}$$

(2) $\vec{r}_2(t) = (\sqrt{1-t^2}, -t)$, $-1 \le t \le 1$

$$\int_C f\,ds = \int_{-1}^{1} \sqrt{1-t^2} \sqrt{\left(\tfrac{d}{dt}\sqrt{1-t^2}\right)^2 + \left(\tfrac{d}{dt}(-t)\right)^2} \, dt$$

$$= \cdots = \int_{-1}^{1} dt = 2 \qquad \text{(check!)}$$

This verifies the fact that the line integral is indep. of the parametrization. (Note: $\vec{r}_1(t)$ & $\vec{r}_2(t)$ are in opposite directions, see later discussion.)



= (signed) area under the graph of f over C. #

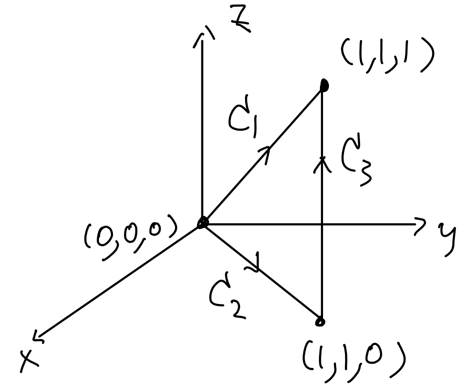
Prop 7: If C is a piecewise smooth curve made by joining $C_1, C_2, \cdots C_n$ end-to-end, then $$\int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds$$

(Pf: Clear from the remark (3) of Def 9', but C_i can be piecewise differentiable in this Prop.)

Remark: "end-to-end" means "end point of C_{k-1} = initial point of C_k".

eg34: Let $f(x,y,z) = x - 3y^2 + z$ (again)

C_1, C_2, C_3 are (straight) line segments as in the figure



We already did $\int_{C_1} f\,ds = 0$

One can similarly calculate

$$\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2} \qquad \text{(Ex!)}$$

(For instance, $\int_{C_3} f\,ds = \int_0^1 (1 - 3(1)^2 + t)\,dt$)

The observation is $\int_{C_1} f\,ds = 0 \neq \int_{C_2 \cup C_3} f\,ds$

even C_1 & $C_2 \cup C_3$ have the same beginning and end points!

(Remark: different from 1-variable calculus)

Conclusion: Line integral of a function depends, not only on the end points, but also the path.