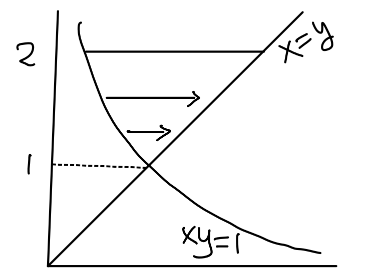
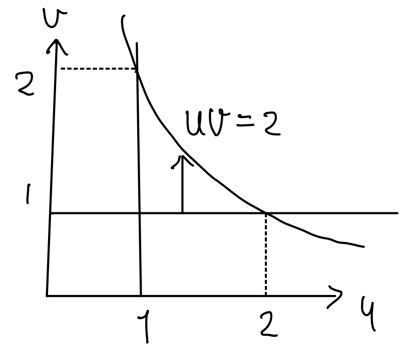
eg30 $I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\,dy$



Soln: Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$

Express x, y in terms of u, v

$$\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$$



Then $\begin{cases} x = y \longleftrightarrow v = 1 \\ x = \frac{1}{y} \longleftrightarrow u = 1 \\ y = 2 \longleftrightarrow uv = 2 \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{2u}{v}$$

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\,dy$$

$$= \int_1^2 \int_1^{\frac{2}{u}} v e^{u} \left|\frac{2u}{v}\right| dv\,du \quad \left(\text{or } \int_1^2 \int_1^{\frac{2}{v}} v e^{2u} \left|\frac{2u}{v}\right| du\,dv\right)$$

$$= \int_1^2 \int_1^{\frac{2}{u}} 2u e^{u}\, dv\,du$$

$$= \int_1^2 2u e^{u} \left(\int_1^{\frac{2}{u}} dv\right) du$$

$$= \int_1^2 2u e^{u} \left(\frac{2}{u} - 1\right) du$$

$$= 2e(e-2) \qquad \text{(check!)}$$

#

eg18 (revisit) Volume of Ellipsiod

$$D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \qquad (a, b, c > 0)$$

$$\mathrm{Vol}(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

Soln Change of variables

$$\begin{cases} u = \frac{x}{a} \\ v = \frac{y}{b} \\ w = \frac{z}{c} \end{cases} \longleftrightarrow \quad \begin{matrix} x = au \\ y = bv \\ z = cw \end{matrix}$$

"New" domain in (u, v, w)-space is

$$G = \{ (u, v, w) : u^2 + v^2 + w^2 \le 1 \}$$

which is the unit ball in (u, v, w)-space.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = abc$$

$$\therefore \quad \mathrm{Vol}(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dw\, dv\, du$$

$$= abc \cdot 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw\, dv\, du$$

$$= abc \ \mathrm{Vol}(\text{Solid unit ball in } (u,v,w)\text{-space})$$

$$= \frac{4\pi}{3} abc$$

#

eg31 Let $D = \{ (x,y,z) \in \mathbb{R}^3 : |x| + |y| + |z| \le 1 \}$

Evaluate $\iiint_D (x+y+z)^4 \, dV$.

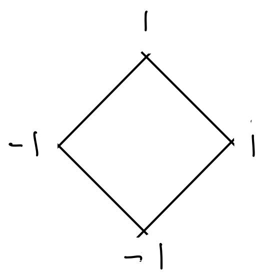
[Remarks: (1) One may use symmetry $(x,y,z) \Leftrightarrow (-x,-y,-z)$ to reduce half, but not to the 1st octant using reflections, since for instance,

$$x+y+z \longleftrightarrow x+y-z \quad \text{under } (x,y,z) \leftrightarrow (x,y,-z),$$

$\therefore (x+y+z)^4$ is not symmetric in all reflections with respect to the coordinates lines

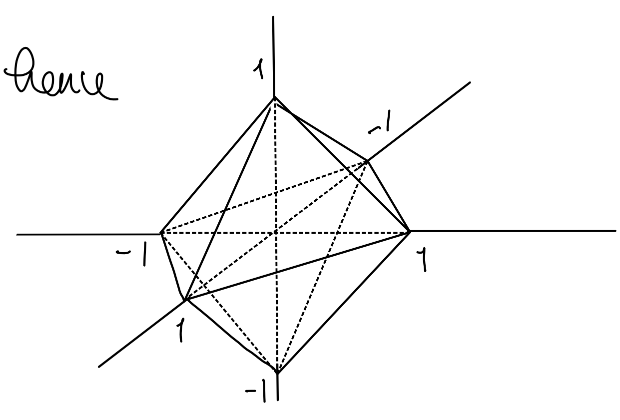
(2) Of course, one may calculate directly without changing variables (Ex!)]

Soln If $z=0$, then $|x|+|y| \le 1$



boundary lines are $\begin{cases} x+y = \pm 1 \\ x-y = \pm 1 \end{cases}$

Similarly for $x=0$ & $y=0$, hence



Boundray surfaces (or faces, or planes) are

$$\begin{cases} x+y+z = \pm 1 \\ x+y-z = \pm 1 \\ x-y-z = \pm 1 \\ x-y+z = \pm 1 \end{cases}$$

Let

$$\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases} \qquad \text{———} (*)$$

Then $\begin{cases} -1 \leq u \leq 1 \\ -1 \leq v \leq 1 \\ -1 \leq w \leq 1 \end{cases}$ & remaining pair becomes $x-y+z = \pm 1$ $u-v+w = \pm 1$

a change of variables formula $\Rightarrow$

$$\iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1 \leq u,v,w \leq 1 \\ -1 \leq u-v+w \leq 1}} u^4 \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dv\,dw\,du$$

By solving $(*)$

$$\begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases} \qquad \text{(check!)}$$

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \left| \det \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \right| = \left| -\frac{1}{4} \right| = \frac{1}{4} \qquad \text{(check!)}$$

Hence $$\iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1 \leq u,v,w \leq 1 \\ -1 \leq u-v+w \leq 1}} \frac{u^4}{4} \, dv\,dw\,du$$

$$= A - B - C$$

where $A = \displaystyle\iiint_{-1 \le u,v,w \le 1} \frac{u^4}{4}\, dv\, dw\, du$

$$B = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \ge 1}} \frac{u^4}{4}\, dv\, dw\, du$$

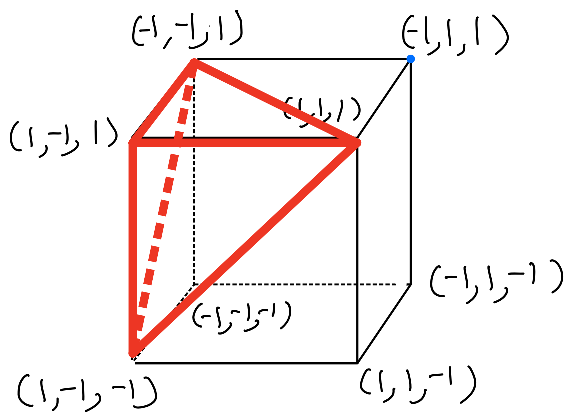
$$C = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \le -1}} \frac{u^4}{4}\, dv\, dw\, du$$

It is clear that

$$A = \iiint_{-1 \le u,v,w \le 1} \frac{u^4}{4}\, dv\, dw\, du = \frac{2}{5} \quad \text{(easy ex!)}$$

To handle B (& C)

	$u-v+w$
$(1,-1,1)$	3
$(1,1,1)$	1
$(-1,-1,1)$	1
$(1,-1,-1)$	1

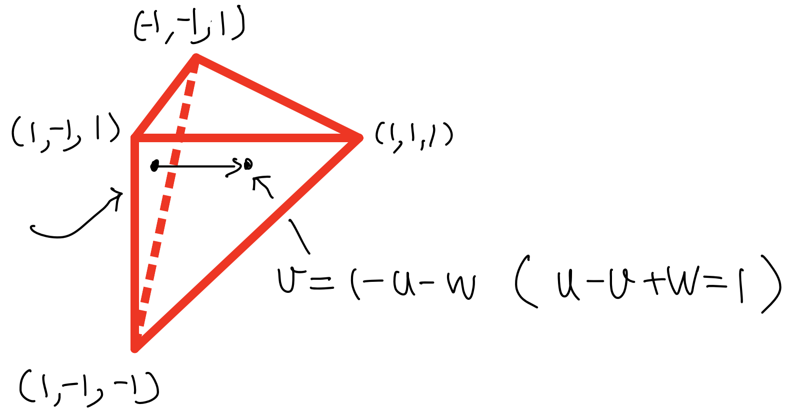


Hence the 3 pts $(1,1,1)$, $(-1,-1,1)$, $(1,-1,-1)$ are on the boundary plane $u-v+w=1$ of B.

Since plane determined by 3 pts, $u-v+w=1$ is the plane passing thro. these 3 pts.

So the solid of integration B, $\left\{\begin{array}{l} -1\le u, v, w\le 1 \\ u-v+w\ge 1 \end{array}\right\}$

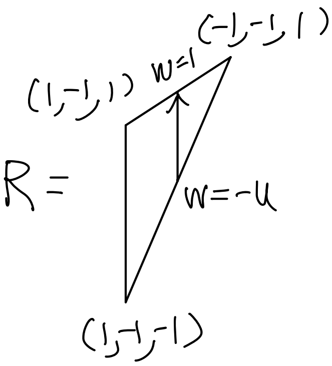
is



$\{v=-1\}$
$=\{(u,-1,w)\}$

which is of special type:

$$B=\iint_R\left(\int_{-1}^{1-u-w}\frac{u^4}{4}\,dv\right)dw\,du, \quad \text{where } R=$$



$$=\int_{-1}^{1}\left(\int_{-u}^{1}\left[\int_{-1}^{1-u-w}\frac{u^4}{4}\,dv\right]dw\right)du$$

$$=\cdots=\frac{3}{35} \quad \text{(check!)}$$

Symmetry (similarly) $\Rightarrow C=B=\frac{3}{35}$

(The solid for the integration C is determined by the 4 pts $(-1,1,-1)$, $(-1,-1,-1)$, $(1,1,-1)$ & $(-1,1,1)$)

Finally $\displaystyle\iiint_D (x+y+z)^4\,dV = A-B-C = \frac{2}{5}-2\cdot\frac{3}{35}=\frac{8}{35}$

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