Prop 4: (a) If $f(x,y) \geq 0$ is an integrable function on a closed (& bdd) rectangle R, then

$$\iint_R f(x,y)\,dA \geq 0 .$$

(b) If R_1 and R_2 be two closed (& bdd) rectangles such that $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$, then

$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

for integrable function over $R_1 \cup R_2$.

Pf: Omitted (Obvious from the concept of Riemann sum)

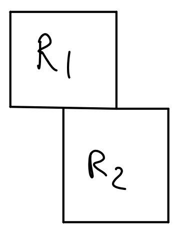
Note: Various situations for $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$

(1)

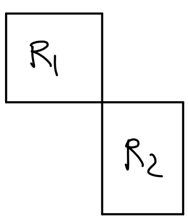
$R_1 \cap R_2 =$ common edge

$\operatorname{int} R_1 \cap \operatorname{int} R_2 = \emptyset$

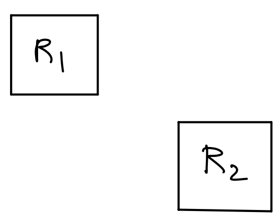
(2)



(3)



(4)

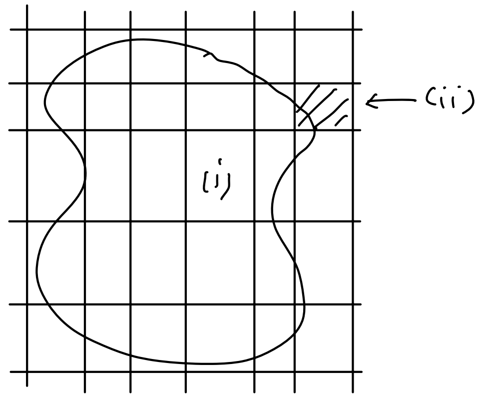


We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2) - (4) !

Hence we need to define double integrals over general regions.

Double Integrals over General Regions

For non-rectangular bounded (closed) region R, one can define similarly the concept of "Riemann sum".

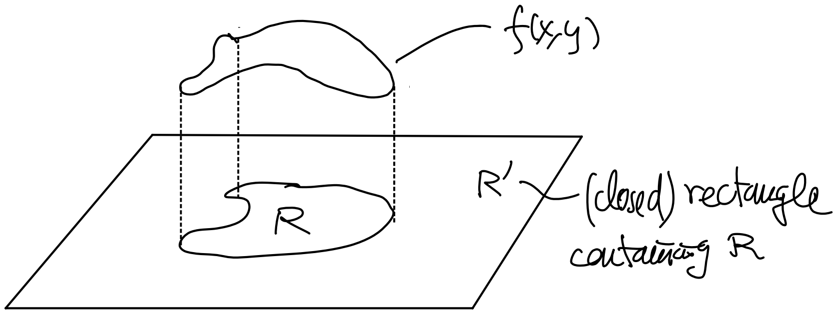


There are two ways to form the "sum"

(i) sum over all subrectangles completely inside R

(ii) sum over all subrectangles with non-empty intersection with R.

Or, one can define "the integrals" as follows



Def 2 : Let R be a bounded region and $f(x,y)$ be a function defined on R. For any rectangle $R' \supset R$, define

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R' \setminus R \end{cases}$$

Then the integral of f over R is defined by

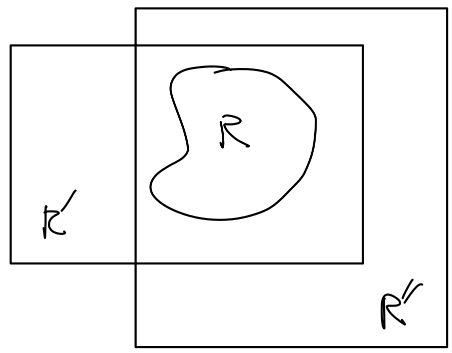
$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

Remark: The definition is well-defined (i.e. doesn't depend on the choice of R'): If R'' is another rectangle s.t. $R'' \supset R$ and

$$\tilde{F}(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R'' \setminus R \end{cases}$$

Then $\displaystyle\iint_{R''} \tilde{F}(x,y)\, dA = \iint_{R'} F(x,y)\, dA$

(by Prop 4 (b))



> **Prop 5**: The propositions 1–4 hold if we replace "closed rectangle" by "closed and bounded region"

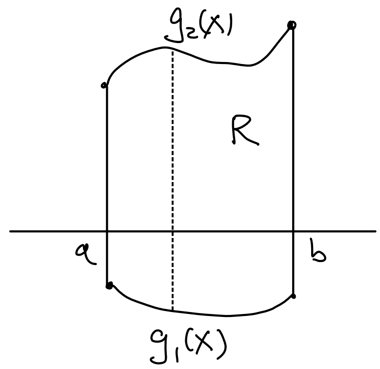
(together with the Prop 2')

Important special types of bounded regions R

Type (1): $R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$

where g_1 and g_2 are "continuous" functions on $[a,b]$.

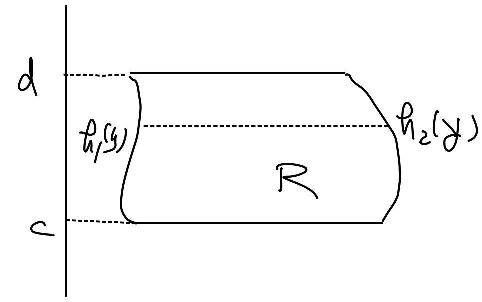
($g_1 \leq g_2$, but $g_1 \not\equiv g_2$)



Type (2) : $R = \{(x,y) : h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$

($h_1 \leq h_2$, but $h_1 \not\equiv h_2$)



For these 2 types of bounded regions, we have

Thm 2 (Fubini's Thm (Stronger version))

Let $f(x,y)$ be a continuous function on a closed and bounded region R.

(1) If R is of type (1) as above, then

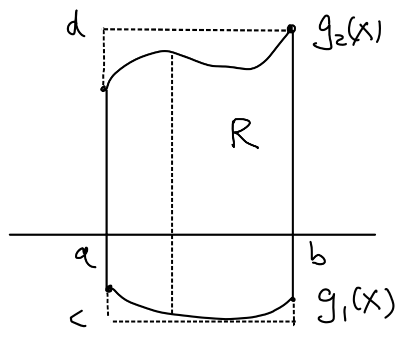
$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Pf: Type (1) : Extend $f(x,y)$ to $F(x,y)$ as in the definition on the rectangle $R' = [a,b] \times [c,d]$ such that

$$c = \min_{[a,b]} g_1(x), \quad d = \max_{[a,b]} g_2(x)$$



By definition 2, $\displaystyle\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$

$$= \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx \qquad \left(\text{Fubini } (1^{st} \text{ form})\right)$$

[f continuous on $R \Rightarrow F$ continuous on R' except possibly on the boundary (curve(s)) of R. Hence by Prop 2', F (in fact $|F|$) is integrable over R'. And the Fubini theorem (1^{st} form) is in fact true for "absolutely" integrable functions on a rectangle.]

Now $F(x,y) = 0$ for $y < g_1(x)$ and $y > g_2(x)$,

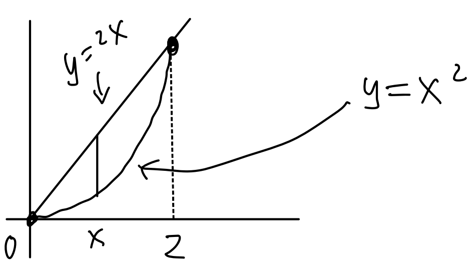
and $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

$$\therefore \iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx.$$

Type (2) can be proved similarly. ※

eg 7 Integrate $f(x,y) = 4y+2$
over the region bounded by $y = x^2$ and $y = 2x$.

Soln:



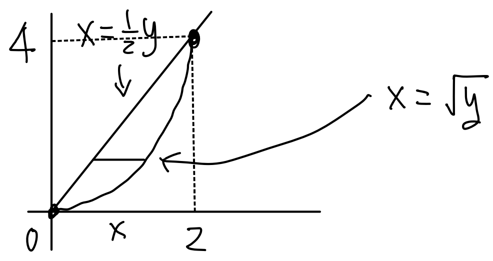
is type (1): $R = \{0 \le x \le 2,\ x^2 \le y \le 2x\}$

$$\iint_R f(x,y)\,dA = \int_0^2 \left(\int_{x^2}^{2x} f(x,y)\,dy \right) dx = \int_0^2 \int_{x^2}^{2x} (4y+2)\,dy\,dx$$

$$= \int_0^2 (-2x^4 + 6x^2 + 4x)\,dx \qquad \text{(check!)}$$

$$= \frac{56}{5}$$

Or



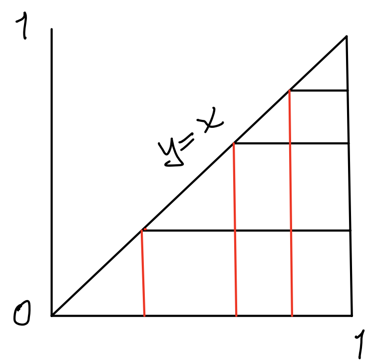
(Also type (2))

$$\iint_A f(x,y)\,dA = \int_0^4 \left(\int_{\frac{1}{2}y}^{\sqrt{y}} (4y+2)\,dx \right) dy$$

$$= \int_0^4 (4y+2)(\sqrt{y} - \tfrac{1}{2}y)\,dy$$

$$= \frac{56}{5} \quad \text{(check!)}$$

eg8: Evaluate $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$.



Soln: Regard $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$

as a double integral of $\frac{\sin x}{x}$

over the region $R = \{ y \le x \le 1,\ 0 \le y \le 1 \}$

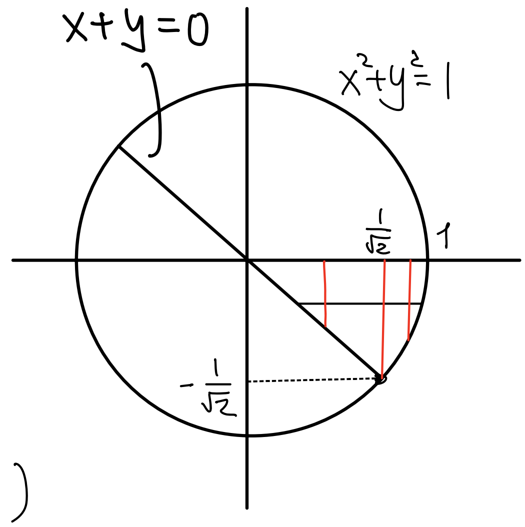
By Fubini's Thm $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy = \iint_R \frac{\sin x}{x}\,dA$

$$= \int_0^1 \int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \sin x\,dx = 1 - \cos 1$$

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$. Why Fubini applied?

(Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$))

eg 9: Find $\iint_R x\,dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$, and the unit circle.



Soln: By Fubini's Thm

$$\iint_R x\,dA = \int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x \, dx\, dy$$

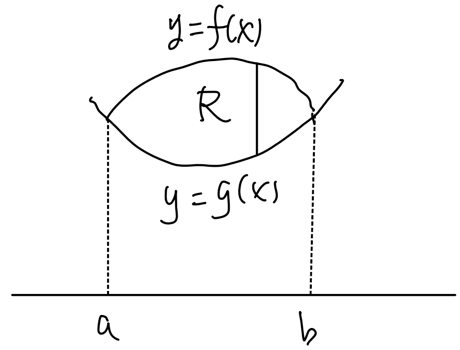
$$= \int_{-\frac{1}{\sqrt{2}}}^{0} \left(\tfrac{1}{2}-y^2\right)dy = \frac{1}{3\sqrt{2}} \text{ (check!)}$$

Alternatively $$\iint_R x\,dA = \int_0^{\frac{1}{\sqrt{2}}} \left(\int_{-x}^{0} x\,dy\right)dx + \int_{\frac{1}{\sqrt{2}}}^{1} \left(\int_{-\sqrt{1-x^2}}^{0} x\,dy\right)dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\,dx = \frac{1}{3\sqrt{2}} \text{ (check!)}$$

Applications

(1) Area (of "good" region $R \subset \mathbb{R}^2$)



$$\boxed{\text{Def 3}: \quad \text{Area}(R) = \iint_R 1\,dA}$$

Then Fubini's Thm implies the well-known formula

$$\text{Area}(R) = \int_a^b [f(x)-g(x)]\,dx \quad (\text{Ex!})$$

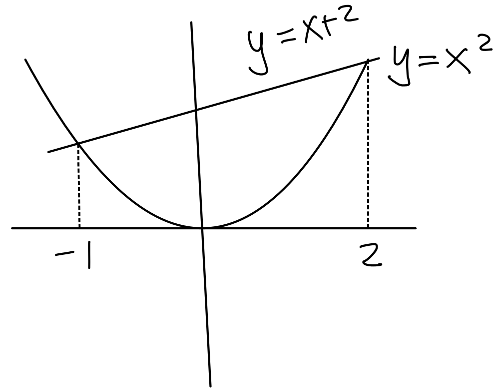
if R is the region bounded by the curves $y=f(x)$ and $y=g(x)$ for $a \le x \le b$

$\left(\text{with } \begin{cases} f(a)=g(a), \\ f(b)=g(b), \\ g(x) \le f(x) \end{cases}\right)$

eg10 Area bounded by $y=x^2$ and $y=x+2$

Soln: Solve $\begin{cases} y=x^2 \\ y=x+2 \end{cases}$

$\Rightarrow \quad x=-1, 2$



Then Fubini's Thm

$$\Rightarrow \text{Area} = \int_{-1}^{2} (x+2-x^2)\,dx = \frac{9}{2} \quad (\text{check!})$$

(2) Average (of a function over a region)

Let $f: R \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function

> Def4: The average value of f over R
> $$= \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$$

eg11 Let $f(x,y) = x\cos(xy)$, $R=[0,\pi]\times[0,1]$

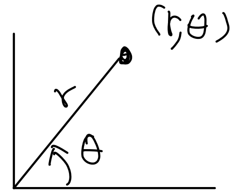
Find average of f over R.

Soln: Average of f over R $= \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

$$= \frac{1}{\pi}\int_0^{\pi}\int_0^1 x\cos(xy)\,dy\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi} \quad (\text{check!})$$

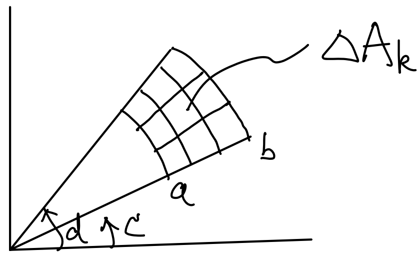
Double integral in polar coordinates



$(r,\theta) \longleftrightarrow (x,y)$

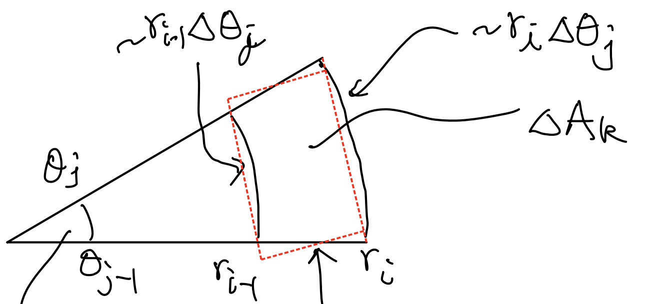
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases}$$



<u>Idea</u>: $\sum_k f(\text{point}_k)\,\Delta A_k$

What is ΔA_k (approximately)?



$\Delta r_i = r_i - r_{i-1}$

$\Delta\theta_j = \theta_j - \theta_{j-1}$

$\therefore \quad \Delta A_k \simeq (r_i \Delta\theta_j)\cdot \Delta r_i \quad \left(\simeq (r_{i-1}\Delta\theta_j)\cdot \Delta r_i\right)$

Hence $\quad \Delta A_k \cong \Delta x\,\Delta y \simeq (r\Delta\theta)\cdot\Delta r$

So $\quad \iint_R f(x,y)\,dA = \iint_R f(x,y)\,\underbrace{dx\,dy}$

$$= \iint_R f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}$$

(length = radius · angle)

Method to remember the formula

$$dA = dx\,dy = r\,dr\,d\theta$$