MATH2230A Tutorial 2

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1. Subsets that preserve (distance) structures

Definitions

Functions: Domain and Rules
(different domain may have different properties)

Let $S \subset \mathbb{C}$. Then

1. We call S open if for all $z \in U$, there exists $r > 0$ such that the open ball $B(z, r) := \{w \in \mathbb{C} : |w - z| < r\}$ centered at z with radius r lies in S
   neighbourhood of z
   open: capture nearness --> limit --> differentiation
2. We call S closed if its complement is open.
3. The smallest closed set containing S is called its closure and is denoted by $\overline{S}$ while the largest open set contained in S is called its interior and is denoted by S^{o}.
4. We call S bounded if $S \subset B(0, r)$ for some $r > 0$.
5. We call S compact if S is closed and bounded.
   path-connected: operations on paths exists --> useful for doing integrations
6. We call S is connected, or path-connected, if for all $z, w \in S$, there exists a continuous curve (function) $\gamma : [0, 1] \to S$ such that $F(0) = z$ and $F(1) = w$, that is, connecting z, w.
7. We call S simply-connected, if S is path-connected and any two continuous curves can be continuously deform to another (or intuitively S is path-connected and has "no holes").

Basic Properties and Example

Exercise 1

Let U_1, U_2 be open sets. Show that

1. $U_1 \cap U_2$ is open.
2. $U_1 \cup U_2$ is open.

Solution of 1.1

Let $x \in U_1 \cap U_2$. Then $x \in U_1$ and $x \in U_2$.
By definition, there exists $r_1, r_2 > 0$ such that $B(x, r_1) \subset U_1$ and $B(x, r_2) \subset U_2$.
Let $r = \min r_1, r_2$. Then $B(x, r) \subset U_1 \cap U_2$

Solution of 1.2

Let $x \in U_1 \cup U_2$. Then $x \in U_1$ or $x \in U_2$.
If $x \in U_1$, by definition, there exists $r_1 > 0$ such that $B(x, r_1) \subset U_1$
Let $r = r_1$. Then $B(x, r) \subset U_1 \subset U_1 \cup U_2$
The case for $x \in U_2$ is similar.

Basic Properties and Example

Exercise 2

We call a subset $K \subset \mathbb{C}$ convex if for all $z, w \in K$ and $t \in [0, 1]$ we have $tz + (1 - t)w \in K$.

1. Show that $B(0, 1)$ is convex. Hint: By Triangle Inequality
2. Show that every convex set is path-connected. Hint: It is direct.

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Real and Imaginary Parts

Definition (Real and Imaginary Parts)

Let $f : U \to \mathbb{C}$ be a function. Then we call the functions $\mathrm{Re}(f) : U \to \mathbb{R}$ and $\mathrm{Im}(f) : U \to \mathbb{R}$ defined by $\mathrm{Re}(f)(z) = \mathrm{Re}(f(z))$ and $\mathrm{Im}(f)(z) = \mathrm{Im}(f(z))$ the real and imaginary part of f respectively.

Polynomials and Rational Functions

Definition (Polynomial Functions)

Let $n \in \mathbb{N}$. Let $a_0, \ldots, a_n \in \mathbb{C}$ with $a_n \neq 0$. We call the function $P_n : U \to \mathbb{C}$ a polynomial function of degree n if it is defined by

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

We call further $a_0, \ldots, a_n$ the coefficients of P_n.

Definition (Rational Functions)

Let $P, Q : U \to \mathbb{C}$ be polynomial functions. Suppose Q is non-zero on U. The quotient $\frac{P}{Q}$ is well-defined and we call it a rational function.

Exponential Function

Definition

Let $z \in \mathbb{C}$. Then we define $e^z := e^x e^{iy}$ if $z = x + iy$ for $x, y \in \mathbb{R}$. Note that e^{iy} is further defined as $\cos y + i \sin y$ where $y \in \mathbb{R}$ by the Euler Formula.

IMPORTANT!!!

The exponential function is NOT injective!

Logarithmic Function

exponential not injective => there is no unique inverse, but inverses exist.

Definition (Complex Logarithms) One Branch, One Inverse

Let $a_0 \in \mathbb{R}$. We call the interval $(a_o, a_0 + 2\pi]$ a branch. We call the function $\log : \mathbb{C}\backslash\{0\}$ defined by $\log z := \ln|z| + i \arg z$, where $\arg z \in (a_o, a_0 + 2\pi]$, the logarithmic function with respective to the branch $(a_o, a_0 + 2\pi]$.

IMPORTANT!!!

If a branch is not chosen, $\log z$ represents a set.

Exercise - Power Functions

Definition (Power functions)

Let $a_0 \in \mathbb{R}$ and $(a_o, a_0 + 2\pi]$ a branch. Let $c \in \mathbb{C}$. We can define $z^c := e^{c \log z}$. The function $z \mapsto z^c$ is called a power function with index c, which is defined on $\mathbb{C} \backslash \{0\}$

Exercise 3

Consider the principle branch. Compute the value of the following:

1. $\log\left(-1 + \sqrt{3} i\right)$
2. i^i
3. $(1 + i)^i$

$(z^a)^b \neq z^{ab}$ in general

$(zw)^a \neq z^a w^a$

Exercise - Trignometric Function Functions

Definition (Trigonometric Functions)

We can define the trigonometric and hyperbolic functions using the exponential functions for all $z \in F$ as follows:

a) $\cos z := \frac{e^{iz}+e^{-iz}}{2}$

b) $\sin z := \frac{e^{iz}-e^{-iz}}{2i}$

Exercise 4

1. Show that $\sin^2 z + \cos^2 z = 1$ for all $z \in \mathbb{C}$
2. Solve $\cos z = 1$.

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2. Functions that preserve (distance structures)

Definitions

Definition

Let $f : U \to \mathbb{C}$ be a function. We say f is continuous at $z_0 \in U$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ if $z \in U$ and $|z - z_0| < \delta$.

iff $f(z) \in B(f(z_0), \varepsilon)$

iff $z \in B(z_0, \delta)$

Theorem

Let $f : U \to \mathbb{C}$ be a function. Let $z_0 \in U$ Then the following are equivalent:

1. *f is continuous at z_0*
2. $\operatorname{Re} f$ *and* $\operatorname{Im} f$ *are continuous at z_0*
3. *$f(z_n) \to f(z_0)$ for all sequence $z_n \in U$ such that $z_n \to z_0$*

Definition

We call $f : U \to \mathbb{C}$ a continuous function if it is continuous for all $z_0 \in U$.

Algebraic Properties of Continuous Functions

Denote $C(U)$ the space of continuous functions from U to $\mathbb{C}$. Then $C(U)$ satisfies the following:

1. $f + g \in C(U)$ if $f, g \in C(U)$
2. $fg \in C(U)$ if $f, g \in C(U)$
3. $kf \in C(U)$ if $f \in C(U), k \in \mathbb{C}$
4. $\frac{f}{g} \in C(U)$ if $f, g \in C(U)$ and g is nonzero on U.
5. If $U = \mathbb{C}$, then $g \circ f \in C(U)$ if $f, g \in C(U)$

The first three shows that the space of continuous functions is a $\mathbb{C}-$ algebra.

Basic Examples

Exercise 5

We are considering the continuity of the conjugate operation. Define $f : \mathbb{C} \to \mathbb{C}$ by $z \mapsto \bar{z}$. Show that

(i). f is continuous

(ii). Hence, the functions $z \mapsto \text{Re}(z)$, $z \mapsto \text{Im}(z)$, $z \mapsto |z|^2$ on the whole space.

Basic Examples

Definition

Let $f : U \to \mathbb{C}$ be a function. We say f is continuous at $z_0 \in U$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ if $z \in U$ and $|z - z_0| < \delta$.

Which of the following are continuous on the domain in which it is defined?

1. Constant Functions
2. Identity Functions
3. Polynomial Functions
4. Exponential Functions
5. Trigonometric Functions
6. Logarithmic Functions (with a chosen branch) ✗
7. Power Functions (with a chosen branch) ✗ depends on the index.

Defined on whole space; continuous on whole space.

Consider Log: Defined on C\{0}. Continuous except on the -ve real axis

Some More Facts

Theorem

Let $f : U \to \mathbb{C}$ be a continuous function. Then we have

1. *$f(U)$ is connected if U is connected.*
2. *$f(U)$ is compact (closed and bounced) if U is compact.*

Corollary (Extreme Value Theorem)

Let $f : U \to \mathbb{C}$ be a continuous function from a closed and bounded (compact) domain U. Then we have
$\sup f(U) = \max f(U)$ *and* $\max f(U) < \infty$

Thank you!
The next Tutorial onwards will be conducted by Kaihui, another TA. Please pay attention to Blackboard Annoucement.