MATH2040A Week 5 Tutorial Notes

1 Invertibility and Isomorphism

A linear map $T \in L(V, W)$ is invertible if there exists another map $U : W \to V$ (between sets) such that $TU = \mathrm{Id}_W$ and $UT = \mathrm{Id}_V$. As shown in lecture, the inverse must also be linear. In another words, the following diagram commutes[1]:

$$\begin{tikzcd} V \arrow[loop left, "\mathrm{Id}_V"] \arrow[rr, bend left, "T"] & & W \arrow[ll, bend left, "U"] \arrow[loop right, "\mathrm{Id}_W"] \end{tikzcd}$$

In such case, we say that V, W are *isomorphic*.

Recall that a linear map preserves the linear structure. Two vector spaces being isomorphic means that the two linear structure can be identified by the other, and they have "the same linear structure".

To show that two finite dimensional vector spaces are isomorphic, instead of finding an isomorphism, you can also use the following result proven in lecture:

Theorem 1.1. *Two* finite dimensional *vectors spaces (over the same scalar field) are isomorphic if and only if they have the same dimension*

Recall the dimension theorem:

Theorem 1.2 (Dimension theorem)**.** *If V is finite dimensional, and $T \in L(V, W)$. Then* $\dim(V) = \dim(\mathsf{N}(T)) + \dim(\mathsf{R}(T))$.

Essentially, the proof of this theorem in lecture is done by showing that the following result:

Lemma 1.3. *Under the same assumptions of dimension theorem, if $\{ v_1, \dots, v_m \}$ is a basis of $\mathsf{N}(T)$ and $\{ v_1, \dots, v_m, v_{m+1}, \dots, v_n \}$ is a basis of V, then $\{ Tv_{m+1}, \dots, Tv_n \}$ is a basis of $\mathsf{R}(T)$.*

I am not aware if this is specifically mentioned in lecture, but this turns out to be quite handy in constructing linear maps. In fact, you can go a bit further with the same proof:

Theorem 1.4. *Under the same assumptions of dimension theorem, every complement[2] R to $\mathsf{N}(T)$ in V is isomorphic with $\mathsf{R}(T)$, with $T|_R : R \to \mathsf{R}(T)$ being an isomorphism.*

Using the lemma with the result that injective linear map preserves linear independence (textbook Sec. 2.1 Q14, homework 4), it is immediate to see the linear independence of certain vectors in $\mathsf{R}(T)$. The following easy result from textbook Sec. 2.1 Q13 is also quite handy when working in the reverse direction:

Lemma 1.5. *Let $T \in L(V, W)$ and $w_1, \dots, w_n \in \mathsf{R}(T)$ be linearly independent. If $v_1, \dots, v_n \in V$ satisfy $Tv_i = w_i$ for each i, then $v_1, \dots, v_n$ are also linearly independent.*

1.1 Coordinate Maps

In lecture, given an ordered basis $\beta = \{ v_1, \dots, v_n \}$ of a finite dimensional vector space V (with $n = \dim(V)$), we know how to convert a vector $v \in V$ to an element $[v]_\beta \in F^n$ in the coordinate space via the combination $v = \sum([v]_\beta)_i v_i$. This in fact gives a linear map $[\cdot]_\beta : V \to F^n$, which we know is linear.

When we are also given an ordered basis $\gamma = \{ w_1, \dots, w_m \}$ of another finite dimensional vector spaces W (with $m = \dim(W)$), we can convert a linear map $T \in L(V, W)$ to a matrix $[T]_\beta^\gamma \in F^{m \times n}$ via the definition $Tv_j = \sum([T]_\beta^\gamma)_{ij} w_i$.

These maps have many properties:

[1]Every directed path that has the same start and same end gives the same result via composition.

[2]Subspace of V such that $V = \mathsf{N}(T) \oplus R$.

- $[Tv]_\gamma = [T]_\beta^\gamma [v]_\beta$
- $[TU]_\alpha^\gamma = [T]_\beta^\gamma [U]_\alpha^\beta$
- T is invertible iff $[T]_\beta^\gamma$ is invertible, with inverse $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$
- $[\cdot]_\beta : V \to F^n$ and $[\cdot]_\beta^\gamma : L(V, W) \to F^{m\times n}$ are isomorphisms on fixed bases β, γ

The coordinate map $[\cdot]_\beta$ has a simple relation with linear combination: for an ordered list $\beta = (v_1, \ldots, v_n)$ of vectors in V, you can show that the linear map $T : F^n \to V$ defined by $T(c_1, \ldots, c_n) = \sum_{i=1}^n c_i v_i$ is injective (resp. surjective) if and only if β is linearly independent (resp. spans V); and so T is an isomorphism if and only if β is a basis, with $T^{-1} = [\cdot]_\beta$.

The invertibility property gives a simple way to check if a linear map $T \in L(V, W)$ between two finite dimensional vector spaces is an isomorphism: just check if its matrix representation $[T]_\beta^\gamma$ is invertible under some bases β, γ. One direct conclusion is the following: if $\beta = \{ v_1, \ldots, v_n \}$ is a basis of a (finite dimensional) vector space V, and $T \in L(V)$ is a linear map such that $T\beta = \{ Tv_1, \ldots, Tv_n \}$ is also a basis, then T is invertible. While this can shown with more elementary argument, we can also observe that the matrix representation of T (with respect to β and $T\beta$) is $[T]_\beta^{T\beta} = I_n$.

1.2 Dual Spaces

Let V be a vector space over scalar field F. Since F is also a vector space over F (with obvious structure), it makes sense to talk about the (algebraic) *dual space* $V^* = L(V, F)$.

If we equip V^* with the (natural) linear structure

- $(T + U)(v) = T(v) + U(v)$ for all $v \in V$
- $(aT)(v) = a \cdot Tv$ for all $v \in V$

It is easy to see that V^* is also a vector space (over F).

We can also consider the *double dual space* $V^{**} = (V^*)^* = L(L(V, F), F)$, which when equipped with a similar linear structure is also a vector space. Concerning these spaces, we have the following properties:

Theorem 1.6. *If V is finite dimensional, $V \cong V^{**}$ and $V \cong V^*$.*

I will omit the detailed proof here, but the basic idea is to

- show that the map $e : V \to V^{**}$ such that, for each $v \in V$, $e(v) \in V^{**}$ maps $f \in V^*$ to $f(v)$, is an isomorphism
- consider a basis $\beta = \{ v_1, \ldots, v_n \}$ and correspondingly a list of mappings $\beta^* = \{ f_1, \ldots, f_n \} \subseteq V^*$ which satisfy $f_i(v_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$, and show that β^* is also a basis

It should be a simple exercise to fill in the blanks.

For $V \cong V^*$, the basis β^* constructed is commonly referred to as the *dual basis*. Effectively, the isomorphism here is constructed as a mapping between the bases β and the dual basis β^*. Due to this dependency of choice of basis β, this isomorphism is (generally) considered to be not *"natural"*, while the isomorphism $e : v \mapsto e(v)$ for $V \cong V^{**}$ is considered "natural".

As an example for this naturality, consider the case where $\dim(V) = 2$, and suppose on a basis $\{v_1, v_2\}$ of V the two isomorphisms give two bases $\{f_1, f_2\}$ of V^* and $\{\varphi_1 = e(v_1), \varphi_2 = e(v_2)\}$ of V^{**}. On another choice $\{v_1, v_1 + v_2\}$ of basis of V, the corresponding basis for V^{**} is $\{\varphi_1, \varphi_1 + \varphi_2\}$, while the corresponding basis for V^* is $\{f_1 - f_2, f_2\}$ instead of $\{f_1, f_1 + f_2\}$.

Unlike most results we have seen so far that continue to hold for infinite dimensional vector spaces (with minor modifications), this *does not hold* if V is infinite dimensional. In such case (under appropriate definition of dimension) we only have $\dim(V) \leq \dim(V^*)$ with strict inequality when V is infinite dimensional. In particular we no longer have isomorphisms between these spaces in general.

2 Exercises

1. Let V, W be finite dimensional vector spaces over F. Let $t : L(V, W) \to L(W^*, V^*)$ be defined by $(t(T))(g) = g \circ T$ (composition) for $g \in W^*$. Show that t is a linear isomorphism.

   (If we denote $t(T)$ as T^*, for each $g \in W^*$ we have $T^*g \in V^*$ such that for each $v \in V$, $(T^*g)(v) = g(Tv)$.)

Solution: We first show that t is linear.

For the sake of notation, let us denote for the moment the application of t as $t(T) = T^*$ for all $T \in L(V, W)$.
Let $T, S \in L(V, W)$. Then for each $g \in W^*$ and $v \in V$, $((T+S)^*g)(v) = g(\,(T+S)v\,) = g(Tv)+g(Sv) = (T^*g)(v)+(S^*g)(v) = (T^*g+S^*g)(v) = ((T^*+S^*)(g))(v)$, so $t(T+S) = (T+S)^* = T^*+S^* = t(T)+t(S)$.
Let $T \in L(V, W)$ and $c \in F$. Then for each $g \in W^*$ and $v \in V$, $((cT)^*g)(v) = g((\,cT\,)\,v) = g(cTv) = c\,(\,g(Tv)\,) = c\,(\,(T^*g)(v)\,) = (\,c(T^*g)\,)\,(v) = ((\,cT^*\,)\,g)(v)$, so $t(cT) = (cT)^* = cT^* = ct(T)$.
So t is linear.

We now show that t is an isomorphism.
We already know that $V \cong V^*$ and $W \cong W^*$, so $\dim(V) = \dim(V^*)$ and $\dim(W) = \dim(W^*)$.
By the result from lecture, we have $\dim L(V, W) = \dim(V)\dim(W) = \dim(W^*)\dim(V^*) = \dim L(W^*, V^*)$.
So by dimension theorem, it suffices to show that t is injective.

Let $T \in \mathsf{N}(\,t\,)$. We want to show that T is the zero vector in $L(V, W)$, that is the zero map.
Let $v \in V$. Suppose $Tv \neq 0$.
Then $\{Tv\} \subseteq W$ is linearly independent, so there exists $g \in L(W, F) = W^*$ such that $(t(T)g)(v) = g(Tv) = 1 \neq 0$.
This implies that $t(T)g \neq 0_{L(V,F)}$, so $t(T) \neq 0_{L(W^*,V^*)}$, contradicting to the assumption that $T \in \mathsf{N}(\,t\,)$.
So $Tv = 0$. As $v \in V$ is arbitrary, $T = 0_{L(V,W)}$.
Therefore t is injective.

Note

If we write the application of a map $f \in V^*$ to $v \in V$ as $f(v) = f^{\mathsf{T}}v$, then the mapping t is defined by the relation $(T^*f)^{\mathsf{T}}v = f^{\mathsf{T}}(Tv)$. When we treat elements in V, W, V^*, W^* as column vectors and linear maps in $L(V, W), L(W^*, V^*)$ as matrices, t just maps a matrix to its transpose. Perhaps for this reason, t is commonly referred to as the *transpose / adjoint map*.

On the other hand, the transpose of a generic vector (not necessarily a linear map) would require some additional structure that will be covered in later lectures.

2. Let V, W be vector spaces (over the same scalar field) such that $V, W \neq \{0\}$ are both nontrivial, and W is finite dimensional. Suppose there exists $T \in L(V, W)$ such that for every $U \in L(W, V)$ that is not the zero map, TUT is also not the zero map. Show that $V \cong W$.

Solution: We will show that $V \cong W$ by showing that T is an isomorphism.

We first show that T is injective. Let $v \in \mathsf{N}(\,T\,)$.
Let $\beta = \{\,w_1, \dots, w_n\,\}$ be a basis of W with $n = \dim(W) \geq 1$, and let $U \in L(W, V)$ be a linear map such that $Uw_i = v$ for each i.
Then for each $x \in V$, $Tx \in W$ and so $Tx = \sum_{i=1}^{n} c_i w_i$ for some $c_1, \dots, c_n$, which implies $TUTx = TU(\sum c_i w_i) = T((\sum c_i)v) = (\sum c_i)Tv = 0$, so TUT is the zero map.
By assumption, this implies that U is also the zero map. As $n \geq 1$, we must have $v = Uw_1 = 0$.
Hence, T is injective.

Suppose that T is not surjective. Then $\mathsf{R}(\,T\,) \subsetneq W$, and so on $m = \dim(\mathsf{R}(\,T\,))$ and $n = \dim(W)$, we must have $m < n$.
Let $\{\,w_1, \dots, w_m\,\}$ be a basis of $\mathsf{R}(\,T\,)$. We can extend it to a basis $\{\,w_1, \dots, w_n\,\}$ of W. As $m < n$,

there is at least one vector $w_{m+1} \notin \mathsf{R}(T)$ that is in this basis of W.
Let $v \in V$ be nonzero, and let $U \in L(W, V)$ be a linear map that satisfies $Uw_i = 0$ for all $i \leq m$ and $Uw_i = v$ for all $i > m$. Since $m < n$ and $Uw_{m+1} = v \neq 0$, there is at least one vector $w_{m+1} \in W$ on which U does not map to the zero vector, so U is not the zero map.
Let $x \in V$. Then $Tx \in \mathsf{R}(T)$ and so $Tx = \sum_{i=1}^{m} c_i w_i$ for some $c_1, \ldots, c_m$, which implies that $TUTx = TU(\sum_{i=1}^{m} c_i w_i) = T(\sum_{i=1}^{m} c_i U w_i) = 0$.
Thus TUT is the zero map. By assumption, this means that U must be the zero map, a contradiction. So T is surjective.

Therefore, T is an isomorphism.

3. Let V be a finite dimensional vector space and $T \in L(V)$. Show that there exists an isomorphism $U \in L(V)$ such that $TUT = T$.

Solution: Let $\{ v_1, \ldots, v_m \}$ be a basis of $\mathsf{N}(T)$ with $m = \text{nullity}(T)$, and extend it to a basis $\{ v_1, \ldots, v_n \}$ with $n = \dim(V) \geq m$.
For each $i \geq m+1$ let $w_i = Tv_i$. Then with the same proof as dimension theorem, $\{ w_{m+1}, \ldots, w_n \}$ is a basis of $\mathsf{R}(T)$. We can extend it to a basis $\{ w_1, \ldots, w_n \}$ of V.
Let $U \in L(V)$ be the linear map that satisfies $Uw_i = v_i$ for each i. Since $\{ v_1, \ldots, v_n \}$ and $\{ w_1, \ldots, w_n \}$ are bases, U is invertible.

We now show that U satisfies the requirement.
For each $i \leq m$, $v_i \in \mathsf{N}(T)$, so $TUTv_i = TU0 = 0 = Tv_i$.
For each $i \geq m+1$, $Tv_i = w_i$, so $TUTv_i = TUw_i = Tv_i$.
So $TUT = T$ on a basis $\{ v_1, \ldots, v_n \}$ of V, and thus $TUT = T$.

Note

Effectively, what we have constructed are the direct sum decompositions $V = \mathsf{N}(T) \oplus R = N \oplus \mathsf{R}(T)$ where $R \cong \mathsf{R}(T)$ and $N \cong \mathsf{N}(T)$, and U can be seen as a combination of two isomorphisms $U|_{\mathsf{R}(T)} : \mathsf{R}(T) \to R$ (which is the inverse of $T|_R : R \to \mathsf{R}(T)$) and $U|_N : N \to \mathsf{N}(T)$ (which we define as a mapping between bases).

4. (Textbook Sec 2.3 Q16)

   Let V, W be finite dimensional vector spaces with $\dim(V) = \dim(W)$, and $T \in L(V, W)$. Show that there exist ordered bases β, γ for V, W respectively such that $[T]_\beta^\gamma$ is diagonal.

Solution: Let $\{ v_1, \ldots, v_m \}$ be a basis of $\mathsf{N}(T) \subseteq V$ where $m = \text{nullity}(T) \leq \dim(V)$, and extend it to a basis $\beta = \{ v_1, \ldots, v_n \}$ of V where $n = \dim(V)$.

With the same proof as dimension theorem, $\{ Tv_{m+1}, \ldots, Tv_n \}$ is linearly independent.

Since $\dim(V) = \dim(W)$, we can extend $\{ Tv_{m+1}, \ldots, Tv_n \}$ to a basis $\gamma = \{ w_1, \ldots, w_m, Tv_{m+1}, \ldots, Tv_n \}$ of W.

We now show that $[T]_\beta^\gamma$ is diagonal.
For each $i \leq m$, $v_i \in \mathsf{N}(T)$, so $Tv_i = 0 = \sum_{j=1}^{m} 0 \cdot w_j + \sum_{j=m+1}^{n} 0 \cdot Tv_j$ with $([Tv_i]_\gamma)_j = 0$ for each j.
For each $i \geq m+1$, $Tv_i = \sum_{j=1}^{m} 0 \cdot w_j + \sum_{j=m+1}^{n} \delta_{ij} \cdot Tv_j$, so $([Tv_i]_\gamma)_j = \delta_{ij}$.
These imply that

$$[T]_\beta^\gamma = \begin{pmatrix} 0_{m \times m} & 0_{m \times (n-m)} \\ 0_{(n-m) \times m} & I_{(n-m) \times (n-m)} \end{pmatrix}$$

which is diagonal.