MATH2040A Week 4 Tutorial Notes

1 Linear Map

A vector space is a set that has some linear structure. A linear map (linear homomorphism) is a mapping between two vector spaces that preserves the linear structure[1]. As a linear structure consists of

- a scalar field
- an addition operation
- a scalar multiplication

this means that for a linear map $T : V \to W$,

- V and W have the same scalar field[2]
- $T(x + y) = T(x) + T(y)$
- $T(ax) = aT(x)$

as defined in the lecture.

As noted in the lecture, a linear map from a finite dimensional vector space is determined solely on its values on some basis: on a basis $\{ v_1, \ldots, v_n \}$ on V, for each given set of vectors $\{ w_1, \ldots, w_n \} \subseteq W$ there exists a unique linear map $T : V \to W$ such that $T(v_i) = w_i$ for each i. By extending a linearly independent to a basis, we can see that the similar conclusion holds on linearly independent set (with choice not necessarily unique).

1.1 Projection

(See textbook Sec 2.1 Q24–27)

Suppose $V = W_1 \oplus W_2$. A map $P = P_{W_1,W_2} : V \to V$ is a projection on W_1 along W_2 if for each decomposition $x = w_1 + w_2 \in V$ with $w_1 \in W_1$, $w_2 \in W_2$ we always have $P(x) = x_1$.

The properties of projection are (see textbook Sec 2.1 Q26 and exercise Q4):

- P is well-defined and is linear
- $\mathsf{R}(P) = W_1$, $\mathsf{N}(P) = W_2$

1.2 Relevant Subspaces

Two subspaces that are the most relevant to the study of linear maps are

- Null space / kernel: $\mathsf{N}(T) = \{ v \in V \mid Tv = 0 \} \subseteq V$
- Range / image: $\mathsf{R}(T) = \{ Tv \mid v \in V \} \subseteq W$
- Invariant subspace (textbook Sec 2.1 Q28–32, assuming $T : V \to V$): $U \subseteq V$ such that $T(U) \subseteq U$

The basic properties are:

- $\mathsf{N}(T)$ and $\mathsf{R}(T)$ are subspaces (of their ambient spaces)

[1] Hence, *homo-morphism*.

[2] This is more of a requirement on the spaces (so that the concept of linear map makes sense) than on the mapping per se.

- T (as a mapping between sets) is injective if and only if $\mathsf{N}(T) = \{0\}$ is trivial
- if β spans V, $T(\beta) = \{ Tv \mid v \in \beta \}$ spans $T(V)$
- (lecture note 6, also textbook Sec 2.1 Q14) if β is linearly independent *and* T is injective, $T(\beta)$ is linearly independent
- (dimension theorem / rank-nullity theorem) assuming V is finite dimensional, $\dim(V) = \dim \mathsf{R}(T) + \dim \mathsf{N}(T) = \operatorname{rank}(T) + \operatorname{nullity}(T)$

Using dimension theorem, you can derive more results. For example, as $\mathsf{R}(T)$ is a subspace of W, (assuming W is finite dimensional) $\mathsf{R}(T) = W$ if they have the same dimension, which you can compute using the dimension theorem.

2 Matrix Representation and Coordinates

Given ordered bases $\beta = (v_1, \ldots, v_n)$, $\gamma = (w_1, \ldots, w_m)$ of (finite dimensional) vector spaces V, W respectively[3],

- the matrix representation / coordinate of a vector $v \in V$ (with respect to β) is the column vector $[x]_\beta = (a_i)_i$ such that
$$x = \sum a_i v_i$$
- the matrix representation of a linear map $T : V \to W$ (with respect to β, γ) is the $\dim(W) \times \dim(V)$ matrix $[T]_\beta^\gamma = (c_{ij})_{ij}$ such that
$$T(v_j) = \sum c_{ij} w_i, \quad \forall j$$

As is shown in the lecture, these representations are linear (on given bases) and have the property that

$$[Tv]_\gamma = [T]_\beta^\gamma [v]_\beta$$
$$[TU]_\alpha^\gamma = [T]_\beta^\gamma [U]_\alpha^\beta$$

To compute such representations, the most straightforward way to do by the definitions:

- To compute $[x]_\beta$, decompose x as a linear combinations of β and write the coefficients as a column vector
- To compute $[T]_\beta^\gamma$, compute each of $[T(v_j)]_\gamma$ for $v_j \in \beta$, then stack the columns of coefficients horizontally

Note that the order of elements in the bases must be kept the same.

2.1 Map and Territory

Do not confuse a vector space with its associated coordinate space, a vector with its coordinate, and a linear map with its matrix representation.

It will soon be proved / is already proven in the lecture that, a n-dimensional vector space over scalar field F has the same linear behavior as F^n. Yet, a (finite dimensional) vector space is *not the same* as its coordinate space, in the same way "a map is *not* the territory": the coordinate space is just a (concrete) representation of the underlying (abstract) vector space in which the linear structure can be easily studied[4]. The precise representation of the (*intrinsic*) linear structure of the vector space via such coordinate space requires an artificial (*extrinsic*) choice of ordered basis, and switching from one choice to another requires some cares (see later lecture about change of coordinate).

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[3]If we *abuse* notation, we can write these definitions in a more compact form:

$$x = \vec{\beta} \cdot [x]_\beta \quad \text{and} \quad \overrightarrow{T\beta} = \vec{\gamma}\,[T]_\beta^\gamma$$

with $\vec{\beta} = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix}$, $\vec{\gamma} = \begin{pmatrix} w_1 & \ldots & w_m \end{pmatrix}$, and $\overrightarrow{T\beta} = \begin{pmatrix} Tv_1 & \ldots & Tv_n \end{pmatrix}$ being "row vectors". Note that these notations *does not really make sense* and only serve as mnemonic devices.

[4]The preservation of linear structure is *too* good that the two spaces are *(linearly) isomorphic*.

3 Exercises

1. Let V, W be vector spaces with $W \neq \{0\}$ nontrivial, and $\{v_1, \dots, v_n\} \subseteq V$ be a linearly dependent subset in V. Show that there exist $w_1, \dots, w_n \in W$ such that no linear map $T \in L(V, W)$ satisfies $T(v_i) = w_i$ for all i.

Solution: Since $\{v_1, \dots, v_n\}$ are linearly dependent, by textbook Sec 1.5 Q15 (homework 2), either $v_1 = 0$, or $v_{k+1} \in \text{Span}(\ \{\ v_1, \dots, v_k\ \}\ )$ for some k.

- If $v_1 = 0$, we can take $w_1 \in W$ to be a nonzero vector, and $w_2 = \dots = w_n = 0$. Then for each $T \in L(V, W)$ we must have $T(v_1) = T0 = 0 \neq w_1$.
- Suppose $v_{k+1} \in \text{Span}(\ \{\ v_1, \dots, v_k\ \}\ )$ for some $k \geq 1$. We may assume that $v_{k+1} = \sum_{i=1}^{k} c_i v_i$ for some scalars $c_1, \dots, c_k$.
  Take $w_{k+1} \in W$ to be a nonzero vector, and $w_1 = \dots = w_k = w_{k+2} = \dots = w_n = 0$. Then for each $T \in L(V, W)$ with $T(v_i) = w_i = 0$ for $i \leq k$, we must have $T(v_{k+1}) = T(\sum_{i=1}^{k} c_i v_i) = \sum_{i=1}^{k} c_i T(v_i) = 0 \neq w_{k+1}$.

In both cases, there exist some $w_1, \dots, w_n \in W$ such that no linear map $T \in L(V, W)$ maps v_i to w_i for each i.

Note

As we can see in the proof, we only need to specify w_1 (or $w_1, \dots, w_{k+1}$), and the remaining vectors do not matter. Still, you should specify them explicitly for they are part of the construction.

2. Let V, W be vector spaces with V finite dimensional, and $T_1, T_2 \in L(V, W)$. Show that $\mathsf{R}(\ T_1\ ) \subseteq \mathsf{R}(\ T_2\ )$ if and only if there exists $S \in L(V, V)$ such that $T_1 = T_2 S$.

Idea: To show that a linear map with such property exists, we just need to construct one concretely, that is to construct a linear map that satisfies $T_1 v = T_2 S v$ for each $v \in V$. Since $\mathsf{R}(\ T_1\ ) \subseteq \mathsf{R}(\ T_2\ )$, we must have $T_1 v = T_2 u$ for some u, so we just need to map $Sv = u$ for each $v \in V$. As V is finite dimensional, we only need this to hold on a basis of V.

Solution: Suppose $T_1 = T_2 S$. Then for each $w \in \mathsf{R}(\ T_1\ )$, $w = T_1 v$ for some $v \in V$, so $w = T_1 v = T_2(Sv) \in \mathsf{R}(\ T_2\ )$. This implies that $\mathsf{R}(\ T_1\ ) \subseteq \mathsf{R}(\ T_2\ )$.

Suppose $\mathsf{R}(\ T_1\ ) \subseteq \mathsf{R}(\ T_2\ )$.
Since V is finite dimensional, V has a basis $\{\ v_1, \dots, v_n\ \}$. As $Tv_i \in \mathsf{R}(\ T_1\ ) \subseteq \mathsf{R}(\ T_2\ )$ for each i, there exist $u_1, \dots, u_n \in V$ such that $T_1 v_i = T_2 u_i$.
Since $\{\ v_1, \dots, v_n\ \}$ is a basis of V, there exists $S \in L(V, V)$ such that $Sv_i = u_i$ for each i.
Then for each i, $(T_2 S)(v_i) = T_2(Sv_i) = T_2 u_i = T_1 v_i$, so $T_1 = T_2 S$ on a basis $\{\ v_1, \dots, v_n\ \}$ of V.
This implies that $T_1 = T_2 S$ (on V).

Note

If $V = \{0\}$, T_1 and T_2 must be the zero map, and the only linear map $S \in L(V, V)$ is also the zero map on S. This is consistent with S constructed in the argument as the basis is an empty set and so we can take arbitrary S (although there is only one linear map).

3. Let V, W be vector space with W finite dimensional, and $T_1, T_2 \in L(V, W)$. Show that $\mathsf{N}(\ T_1\ ) \subseteq \mathsf{N}(\ T_2\ )$ if and only if there exists $S \in L(W, W)$ such that $T_2 = S T_1$.

Idea: To show that a linear map with such property exists, we just need to construct one concretely, that is to construct a linear map that satisfies $T_2 v = ST_1 v$ for each $v \in V$. Since S takes a vector $T_1 v$ from $\mathsf{R}(\ T_1\ )$, we just need S to have the correct images on a basis $\{\ T_1 v_1, \ldots, T_1 v_n\ \}$ of $\mathsf{R}(\ T_1\ )$, that is, to have $S(T_1 v_i) = T_2 v_i$. All that remains is to check that this construction still works when $T_1 v = 0$, which can be done by the kernel condition.

Conceptually this is the same idea as the last one.

Solution: Suppose $T_2 = ST_1$. Then for each $v \in \mathsf{N}(\ T_1\ )$, $T_1 v = 0$, so $T_2 v = ST_1 v = S0 = 0$, $v \in \mathsf{N}(\ T_2\ )$. This implies that $\mathsf{N}(\ T_1\ ) \subseteq \mathsf{N}(\ T_2\ )$.

Suppose $\mathsf{N}(\ T_1\ ) \subseteq \mathsf{N}(\ T_2\ )$.
Since W is finite dimensional, there exists a basis $\{\ T_1 v_1, \ldots, T_1 v_n\ \}$ of $\mathsf{R}(\ T_1\ )$ where $v_1, \ldots, v_n \in V$.
Let $S \in L(W, W)$ be such that $S(T_1 v_i) = T_2 v_i$ for each i. Such S exists as $\{\ T_1 v_1, \ldots, T_1 v_n\ \} \subseteq W$ is linearly independent, and W is finite dimensional.

We now show that $T_2 = ST_1$.
Let $v \in V$. Since $T_1 v \in \mathsf{R}(\ T_1\ )$, there exists $c_1, \ldots, c_n$ such that $T_1 v = \sum c_i T_1 v_i = T_1(\sum c_i v_i)$, so $T_1(v - \sum c_i v_i) = 0$.
This implies that $v - \sum c_i v_i \in \mathsf{N}(\ T_1\ ) \subseteq \mathsf{N}(\ T_2\ )$, so $T_2(v - \sum c_i v_i) = 0$, $T_2 v = \sum c_i T_2 v_i = \sum c_i ST_1 v_i = ST_1(\sum c_i v_i) = ST_1 v$.
As $T_2 v = ST_1 v$ for all $v \in V$, $T_2 = ST_1$.

Note

If $\mathsf{R}(\ T_1\ ) = \{0\}$, $T_1 = 0$ with $\mathsf{N}(\ T_1\ ) = V$, and so $T_2 = 0$ as well. This means that we can choose arbitrary $S \in L(W, W)$ and still have $T_2 = ST_1$. This is consistent with S constructed in the argument (which can be arbitrarily chosen as the basis is an empty set).

4. (Textbook Sec 2.1 Q26(a, b))

Suppose $T : V \to V$ is a projection on W_1 along W_2, where $V = W_1 \oplus W_2$. Show that

(a) T is linear

(b) $W_1 = \mathsf{R}(\ T\ ) = \{\ v \in V \mid Tv = v\ \}$

(c) $W_2 = \mathsf{N}(\ T\ )$

Solution:

(a) Let $x, y \in V$, $a \in F$.
As $V = W_1 \oplus W_2$, there exist unique $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ such that $x = x_1 + x_2$ and $y = y_1 + y_2$.
Also, $x + y = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2)$ with $x_1 + y_1 \in W_1$ and $x_2 + y_2 \in W_2$, and $ax = a(x_1 + x_2) = (ax_1) + (ax_2)$ with $ax_1 \in W_1$, $ax_2 \in W_2$.
By definition of projection, $T(x) = x_1$, $T(y) = y_1$, so $T(x + y) = x_1 + y_1 = T(x) + T(y)$ and $T(ax) = ax_1 = aT(x)$.
These imply that T is linear.

(b) Let $S = \{\ v \in V \mid Tv = v\ \} \subseteq V$.

Let $v \in W_1$. Then $v = v + 0$ with $v \in W_1$ and $0 \in W_2$, so $Tv = v$, $v \in S$. This implies that $W_1 \subseteq S$.

Let $v \in S$. Then there exist unique $v_1 \in W_1$ and $v_2 \in W_2$ such that $v = v_1 + v_2$. By definition, $v = Tv = v_1 \in W_1$. This implies that $S \subseteq W_1$.

Hence $S = W_1$.

Since $S \subseteq \mathsf{R}(T)$, we have $W_1 \subseteq \mathsf{R}(T)$.
By definition of projection, we also have $\mathsf{R}(T) \subseteq W_1$, so $W_1 = \mathsf{R}(T)$.

(c) Let $v \in W_2$. Then $v = 0 + v$ where $0 \in W_1$ and $v \in W_2$. By definition of projection, $Tv = 0$, so $v \in \mathsf{N}(T)$.

Let $v \in \mathsf{N}(T)$. Then there exist $v_1 \in W_1$ and $v_2 \in W_2$ such that $v = v_1 + v_2$. By assumption, $0 = Tv = v_1$, so $v = v_2 \in W_2$.
These imply that $\mathsf{N}(T) = W_2$.

Note

By part (b), for each $v \in V$, $Tv \in W_1 = \{ x \in V \mid Tx = x \}$ and so $T^2 v = T(Tv) = Tv$ (idempotent).