# MATH2040A Week 3 Tutorial Notes

## 1 Basis and dimension

- A *spanning set* $S$ is a subset of a vector space that spans the whole space: $\text{Span}(\ S\ ) = V$
- A *basis* is a linearly independent spanning set
- If there is a *finite* spanning set, the space is called *finite dimensional*. Otherwise it is *infinite dimensional*

It is shown in lecture that a finite dimensional space has a basis, but note that we have *not* shown if an infinite dimensional vector space also has a basis[1], so you should *not* assume that a basis always exists.[2]

If $\beta$ is a basis of $V$, its size $|\ \beta\ |$ is the *dimension* of the vector space $V$. It is shown in lecture that every basis of a finite dimensional vector space has the same size.

The major results about basis are

- Given a basis of a finite dimensional space, every vector has unique representation as a linear combination of elements from the basis.[3]
- A finite spanning set can be reduced to a (finite) basis (by removing some of its elements)
- (Replacement theorem) If $S$ is a finite spanning set and $L$ is a finite linearly independent set (in the same space), then
  - $|\ S\ | \geq |\ L\ |$
  - you can take elements from $S$ and add them to $L$ to make $L$ a spanning set and have the same number of elements as $S$
- In a finite dimensional space, a (finite) linearly independent set can extend to a basis (by adding elements to it). In particular, a basis of a subspace can be extended to one of the whole space.

A (direct) consequence of replacement theorem is that, in a finite dimensional space $V$,

- if $S$ is a spanning set, $\dim(V) \leq |\ S\ |$
- if $L$ is a linearly independent set, $\dim(V) \geq |\ L\ |$

To show that a set $\beta$ is a basis of some space $V$, typically you need show that $\beta$ is *both* linearly independent *and* spans $V$, which you can apply the approaches covered in the last tutorial. However, if you already know the dimension of $V$ (assuming it is finite dimensional), you can just show that $\beta$ has the correct size, and show *either one* of linear independence and span. The issue with this shortcut is that, in many cases, you may need prove why the claimed dimension is correct, and this may require constructing a basis explicitly.

When handling the sum of two subspaces, one common technique is to use the following result[4]:

$$\text{Span}(\ S_1 \cup S_2\ ) = \text{Span}(\ S_1\ ) + \text{Span}(\ S_2\ )$$

If $S_1, S_2$ are basis of $W_1, W_2$ respectively, this implies that $S_1 \cup S_2$ is a spanning set of the subspace $W_1 + W_2$.

[1] This requires axiom of choice, and I do not think we will be discussing it in this course.

[2] But also do not assume that a space is finite dimensional just because it has a basis (e.g. the space of all real polynomials), unless you also know the basis is a finite set.

[3] This actually holds for infinite basis as well: if $v \in V$ and $\beta$ is a (possibly infinite) basis of $V$, then there exist $n \in \mathbb{N}, v_1, \ldots, v_n \in \beta$ and nonzero scalars $c_1, \ldots, c_n \in F$ (which are unique up to permutation of indices) such that $v = \sum c_i v_i$.

[4] See also textbook Sec 1.4 Q14 in homework 2.

### 1.1 A result in lecture note

Here is a result that is mentioned in a remark on the last page of lecture note 4; see also textbook Sec 1.6 Q34(a).

Let $V$ be a finite dimensional vector space (over $F$), and $W$ be a subspace of $V$. Show that there exists a subspace $Q$ of $V$ such that $V = W \oplus Q$.

*Proof.* Since $V$ is finite dimensional, so is $W$. Then $W$ has a basis $\beta = \{ w_1, \ldots, w_n \} \subseteq W$ for some $n \in \mathbb{N}$. By (corollary of) replacement theorem we can extend $\beta$ into a basis $\gamma = \{ w_1, \ldots, w_n, v_1, \ldots, v_m \} \subseteq V$ of $V$ for some $m \in \mathbb{N}$ and $v_1, \ldots, v_m \in V$.

Let $Q = \text{Span}( \beta' )$ with $\beta' = \gamma \setminus \beta = \{ v_1, \ldots, v_m \}$. By property of span, $Q$ is a subspace of $V$. We now show that $V = W \oplus Q$. To do so, we need to check that $W + Q = V$ and $W \cap Q = \{0\}$.

Since $\gamma = \beta \cup \beta'$ is a basis of $V$, $V = \text{Span}( \gamma ) = \text{Span}( \beta \cup \beta' ) = \text{Span}( \beta ) + \text{Span}( \beta' ) = W + Q$.

Trivially, $\{0\} \subseteq W \cap Q$. Let $v \in W \cap Q$. Then there exist scalars $c_1, \ldots, c_n, d_1, \ldots, d_m \in F$ such that $v = \sum_{i=1}^{n} c_i w_i$ and $v = \sum_{j=1}^{m} d_j v_j$. This implies $0 = v - v = c_1 w_1 + \ldots + c_n w_n - d_1 v_1 - \ldots - d_m v_m$. As $\gamma = \{ w_1, \ldots, w_n, v_1, \ldots, v_m \}$ is a basis, it is linear independent, and so $c_1 = \ldots = c_n = d_1 = \ldots = d_m = 0$. Hence $v = \sum_{i=1}^{n} c_i w_i = 0$. As $v \in W \cap Q$ is arbitrary, $W \cap Q = \{0\}$.

Therefore $V = W \oplus Q$. □

### Note

We call such $Q$ a *complement* of $W$. Excluding the trivial case where $W = \{0\}$ or $W = V$, as the way to extend the basis $\beta$ is not unique, the complement is also not unique.

While we consider only finite dimensional vector spaces, this proof *can* be extended to infinite dimensional vector spaces *if* you have the corresponding theorems for infinite dimensional spaces.

## 2 Exercises

1. Let $\{ v_1, \ldots, v_n \}$ be a linearly independent set of vectors in a vector space $V$, and $u \in V$. Show that $\dim \text{Span}( \{ v_1 + u, \ldots, v_n + u \} ) \geq n - 1$.

**Solution:** Let $W = \text{Span}( \{ v_1 + u, \ldots, v_n + u \} )$. Since $\{ v_1 + u, \ldots, v_n + u \}$ is a finite spanning set of $W$, $W$ is finite dimensional.

If $n = 1$, we trivially have that $\dim(W) \geq 0 = n - 1$. Hence, we may assume in the following that $n \geq 2$.

For each $i \in \{ 1, \ldots, n-1 \}$, $v_i - v_n = (v_i + u) - (v_n + u) \in \text{Span}( \{ v_1 + u, \ldots, v_n + u \} ) = W$, so $S = \{ v_1 - v_n, \ldots, v_{n-1} - v_n \} \subseteq W$.
Let $c_1, \ldots, c_{n-1} \in F$ be scalars such that $\sum_{i=1}^{n-1} c_i (v_i - v_n) = 0$. Then $\sum_{i=1}^{n-1} c_i v_i + (\sum_{j=1}^{n-1} c_j) v_n = 0$.
As $\{ v_1, \ldots, v_n \}$ is linearly independent, we must have $c_i = 0$ for all $i \in \{ 1, \ldots, n-1 \}$. So, $\{ v_1 - v_n, \ldots, v_{n-1} - v_n \}$ is linearly independent. In particular, all elements in this set are distinct, so by construction $S$ has $n - 1$ elements.
Since $W$ contains a linearly independent set $S$ of $n - 1$ elements, by (corollary of) replacement theorem $\dim(W) \geq | S | = n - 1$.

2. (See also textbook Sec 1.6 Q31(b))

Let $U_1, \ldots, U_n$ be finite dimensional subspaces of a vector space $V$. Show that $W = U_1 + \ldots + U_n$ is finite dimensional and $\dim(W) \leq \dim(U_1) + \ldots + \dim(U_n)$.

(Here $U_1 + \ldots + U_n = \{ x_1 + \ldots + x_n \mid x_1 \in U_1, \ldots, x_n \in U_n \}$.)

**Solution:** Since $U_1, \ldots, U_n$ are finite dimensional, there exist bases $\beta_1, \ldots, \beta_n$ for $U_1, \ldots, U_n$.
By definition, $\dim(U_1) = | \beta_1 |, \ldots, \dim(U_n) = | \beta_n |$.
By property of span, $W = U_1 + \ldots + U_n = \text{Span}( \beta_1 ) + \ldots + \text{Span}( \beta_n ) = \text{Span}( \beta_1 \cup \cdots \cup \beta_n )$ where $\beta_1 \cup \cdots \cup \beta_n$ is a finite union of finite set and so is finite.

This implies that $W$ is spanned by a finite set and thus finite dimensional, and $\dim(W) \leq |\ \beta_1 \cup \cdots \cup \beta_n\ | \leq |\ \beta_1\ | + \ldots + |\ \beta_n\ | = \dim(W_1) + \ldots + \dim(W_n)$.

## Note

Since we have *only* defined "dimension" for spaces with a finite spanning set (i.e. "finite dimensional" spaces), it is necessary to first check if $W$ is finite dimensional, so that it makes sense to talk about $\dim(W)$.

3. (Textbook Sec 1.6 Q29)

   Prove that if $W_1$ and $W_2$ are finite-dimensional subspaces of a vector space $V$, then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

   Furthermore, assuming $V = W_1 + W_2$, show that this sum is a direct sum if and only if $\dim(V) = \dim(W_1) + \dim(W_2)$.

**Solution:** By the last question, $W_1 + W_2$ is finite dimensional.

Since $W_1 \cap W_2 \subseteq W_1$ is a subspace of a finite dimensional vector space, it is also finite dimensional. Let $\beta = \{\ u_1, \ldots, u_n\ \} \subseteq W_1 \cap W_2$ be a basis of $W_1 \cap W_2$ with $n = \dim(W_1 \cap W_2)$.
Since $W_1 \cap W_2 \subseteq W_1$ and $W_1$ is finite dimensional, we can extend $\beta$ to a basis $\beta_1 = \{\ u_1, \ldots, u_n, v_1, \ldots, v_m\ \}$ of $W_1$, with $m = \dim(W_1) - n \in \mathbb{N}$ and $v_1, \ldots, v_m \in W_1$.
Similarly, extend $\beta$ to a basis $\beta_2 = \{\ u_1, \ldots, u_n, w_1, \ldots, w_p\ \}$ of $W_2$, with $p = \dim(W_2) - n \in \mathbb{N}$ and $w_1, \ldots, w_p \in W_2$.
Let $\gamma = \beta_1 \cup \beta_2 \subseteq W_1 \cup W_2 \subseteq W_1 + W_2$. We will show that $|\ \gamma\ | = n + m + p$ and that $\gamma$ is a basis of $W_1 + W_2$, as these would imply that $\dim(W_1 + W_2) = |\ \gamma\ | = n + m + p = (n + m) + (n + p) - n = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Let $c_1, \ldots, c_n, a_1, \ldots, a_m, b_1, \ldots, b_p \in F$ be scalars such that $\sum_{i=1}^{n} c_i u_i + \sum_{j=1}^{m} a_j v_j + \sum_{k=1}^{p} b_k w_k = 0$. Then $\sum_{i=1}^{n} c_i u_i + \sum_{j=1}^{m} a_j v_j = -\sum_{k=1}^{p} b_k w_k$.
By assumption $\sum_{i=1}^{n} c_i u_i + \sum_{j=1}^{m} a_j v_j \in \text{Span}(\ \beta_1\ ) = W_1$ and $-\sum_{k=1}^{p} b_k w_k \in \text{Span}(\ \beta_2\ ) = W_2$, so $\sum_{i=1}^{n} c_i u_i + \sum_{j=1}^{m} a_j v_j = -\sum_{k=1}^{p} b_k w_k \in W_1 \cap W_2$.
As $\beta$ is a basis of $W_1 \cap W_2$, there exists $d_1, \ldots, d_n \in F$ such that $\sum_{i=1}^{n} c_i u_i + \sum_{j=1}^{m} a_j v_j = \sum_{i=1}^{n} d_i u_i$, which implies $\sum_{i=1}^{n} (c_i - d_i) u_i + \sum_{j=1}^{m} a_j v_j = 0$, with $u_1, \ldots, u_n, v_1, \ldots, v_m \in \beta_1$.
As $\beta_1$ is a basis, it is linearly independent, so we must have $c_1 - d_1 = \ldots = c_n - d_n = a_1 = \ldots = a_m = 0$. In particular, we have $\sum_{j=1}^{m} a_j v_j = 0$.
This implies that $\sum_{i=1}^{n} c_i u_i + \sum_{k=1}^{p} b_k w_k = 0$. As $\beta_2$ is a basis, it is linearly independent and so we must have $c_1 = \ldots = c_n = b_1 = \ldots = b_p = 0$.
It then follows that $\gamma = \{\ u_1, \ldots, u_n, v_1, \ldots, v_m, w_1, \ldots, w_p\ \}$ is linearly independent. In particular, all elements are distinct, and so $|\ \gamma\ | = n + m + p$.

To show that $\gamma$ is a basis of $W_1 + W_2$, it remains to show $\text{Span}(\ \gamma\ ) = W_1 + W_2$.
Since $\gamma = \beta_1 \cup \beta_2$, by property of span we have $\text{Span}(\ \gamma\ ) = \text{Span}(\ \beta_1 \cup \beta_2\ ) = \text{Span}(\ \beta_1\ ) + \text{Span}(\ \beta_2\ ) = W_1 + W_2$.

Therefore, $\gamma$ is a basis of $W_1 + W_2$.

We now show the second part. As we already have $V = W_1 + W_2$, $V = W_1 \oplus W_2$ iff $W_1 \cap W_2 = \{0\}$, which holds iff $\dim(W_1 \cap W_2) = 0$. By our conclusion in the first part, this holds iff $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$.

## Note

If we assume that every vector space has a basis, it is possible to extend the result to infinite dimensional spaces, although we can only have $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim(W_1) + \dim(W_2)$ (as cardinals), as subtraction does not make sense in this context.

4. Determine if the following identity holds on every vector space $V$ and its finite dimensional subspaces $W_1, W_2, W_3$:

$$\begin{aligned}\dim(W_1 + W_2 + W_3) = \dim(W_1) &+ \dim(W_2) + \dim(W_3) \\ &- \dim(W_1 \cap W_2) - \dim(W_1 \cap W_3) - \dim(W_2 \cap W_3) \\ &+ \dim(W_1 \cap W_2 \cap W_3)\end{aligned}$$

(Compare with inclusion-exclusion principle for sets.)

**Solution:** The identity does not hold in general. One counterexample is $V = \mathbb{R}^2$ being the usual real plane, and $W_1 = \{ (x, 0) \mid x \in \mathbb{R} \}$, $W_2 = \{ (0, y) \mid y \in \mathbb{R} \}$, $W_3 = \{ (x, x) \mid x \in \mathbb{R} \}$.
It is easy to check that

- $W_1, W_2, W_3$ are finite dimensional subspaces of (finite dimensional) vector space $V$
- $\dim(W_1) = \dim(W_2) = \dim(W_3) = 1$
- $W_1 \cap W_2 = W_1 \cap W_3 = W_2 \cap W_3 = W_1 \cap W_2 \cap W_3 = \{0\}$ which has dimension $\dim(\{0\}) = 0$
- $V = W_1 + W_2 + W_3$ which has dimension $\dim(V) = 2$

and so on these spaces, LHS evaluates to 2 while RHS evaluates to $1 + 1 + 1 - 0 - 0 - 0 + 0 = 3 \neq 2$.

## Note

This appears to be a (surprisingly) common misbelief.

The counterexample provided here is the same one as in tutorial 1. In fact, using the result of Q3 on $W_1 + W_2$ and $(W_1 \cap W_3) + (W_2 \cap W_3)$ and noting that $W_1 \cap W_2 \cap W_3 = (W_1 \cap W_3) \cap (W_2 \cap W_3)$, we have

$$\begin{aligned}\text{RHS} &= \dim(W_1 + W_2) + \dim(W_3) - \dim((W_1 \cap W_3) + (W_2 \cap W_3)) \\ &= \dim(W_1 + W_2 + W_3) + \dim((W_1 + W_2) \cap W_3) - \dim((W_1 \cap W_3) + (W_3 \cap W_3))\end{aligned}$$

so the identity holds iff

$$\dim((W_1 + W_2) \cap W_3) = \dim((W_1 \cap W_3) + (W_2 \cap W_3))$$

and it is easy to see that $(W_1 + W_2) \cap W_3 \supseteq (W_1 \cap W_3) + (W_2 \cap W_3)$, so this holds iff

$$(W_1 + W_2) \cap W_3 = (W_1 \cap W_3) + (W_2 \cap W_3)$$

which, as shown by the counterexample, does not hold in general.