MATH2040A Week 1 Tutorial Notes

1 Vector space

A vector space is a (nonempty) set that has a linear structure. Linear structure means that you can add two elements together, and scale an element by some scalar multiplier, and they act in the same way you would expect. More precisely, these two operations (addition and scalar multiplication) satisfy the following 8 conditions from lecture:

1. Addition is commutative: $x + y = y + x$
2. Addition is associative: $(x + y) + z = x + (y + z)$
3. Zero vector: there exists $\vec{0}$ such that $x + \vec{0} = x$
4. Additive inverse: for all x there exists y such that $x + y = \vec{0}$
5. Unit scalar multiplication: $1 \cdot x = x$
6. Scalar multiplication is associative: $(\alpha\beta) \cdot x = \alpha \cdot (\beta \cdot x)$
7. Distributive law 1: $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$
8. Distributive law 2: $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$

There is a hidden condition[1] that is sometimes neglected: the two operations must be well-defined. This means that, for example, for all elements x, y in the set, $x + y$ is uniquely defined and *is still in the set*. Most of the time "uniquely defined" is obvious, but the closure part may need some justification.

These properties are the same ones of real number addition and multiplication which you are familiar with. In a sense, a real vector space is an extension of real numbers.

To check if a set (equipped with a scalar field and the two operations) is a vector space, you can do it from the first principle: just verify *all* these 8 conditions one by one.

1.1 Exercises

1. Determine if the set $S = (0, \infty)$ is a real vector space when equipped with the following operations:

   - addition: $x \oplus y = 2xy$ for $x, y \in S$
   - scalar multiplication: $\alpha \odot x = 2^{\alpha-1}x^\alpha$ for $x \in S$, $\alpha \in \mathbb{R}$

   If so, what is its zero vector?

Solution: It is easy to see that $x \oplus y = 2xy \in S$ for all $x, y \in S$ and $\alpha \odot x = 2^{\alpha-1}x^\alpha \in S$ for all $x \in S$ and $\alpha \in \mathbb{R}$. To show that S is a vector space, we verify all 8 conditions one by one:

1. For all $x, y \in S$, $x \oplus y = 2xy = 2yx = y \oplus x$
2. For all $x, y, z \in S$, $(x \oplus y) \oplus z = 2(x \oplus y)z = 2(2xy)z = 4xyz = 2x(2yz) = 2x(y \oplus z) = x \oplus (y \oplus z)$
3. Take $\vec{0} = \frac{1}{2} \in S$. Then for all x, $x \oplus \vec{0} = 2x\frac{1}{2} = x$

[1]Technically two conditions.

4. For all $x \in S$, on $y = \frac{1}{4x} \in S$, $x \oplus y = 2x\frac{1}{4x} = \frac{1}{2} = \vec{0}$
5. For all $x \in S$, $1 \odot x = 2^{1-1}x^1 = x$
6. For all $x \in S$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \odot (\beta \odot x) = \alpha \odot (2^{\beta-1}x^\beta) = 2^{\alpha-1}(2^{\beta-1}x^\beta)^\alpha = 2^{\alpha\beta-1}x^{\alpha\beta} = (\alpha\beta) \odot x$
7. For all $\alpha \in \mathbb{R}$ and $x, y \in S$, $\alpha \odot (x \oplus y) = 2^{\alpha-1}(2xy)^\alpha = 2(2^{\alpha-1}x^\alpha)(2^{\alpha-1}y^\alpha) = (\alpha \odot x) \oplus (\alpha \odot y)$
8. For all $\alpha, \beta \in \mathbb{R}$ and $x \in S$, $(\alpha + \beta) \odot x = 2^{\alpha+\beta-1}x^{\alpha+\beta} = 2(2^{\alpha-1}x^\alpha)(2^{\beta-1}x^\beta) = (\alpha \odot x) \oplus (\beta \odot x)$

As all conditions are satisfied, S is a real vector space with the given operations.

As seen in the proof, its zero vector is $\frac{1}{2}$.

2. Determine if the set $S = \mathbb{R}$ is a real vector space when equipped with the following operations:
   - addition: $x \oplus y = x + y - 2$ for $x, y \in S$
   - scalar multiplication: $\alpha \odot x = \alpha x + 2(1 - \alpha)$ for $x \in S$, $\alpha \in \mathbb{R}$

   If so, what is its zero vector?

Solution: It is easy to see that $x \oplus y = x + y - 2 \in S$ for all $x, y \in S$ and $\alpha \odot x = \alpha x + 2(1 - \alpha) \in S$ for all $x \in S$ and $\alpha \in \mathbb{R}$. To show that S is a vector space, we verify all 8 conditions one by one:

1. For all $x, y \in S$, $x \oplus y = x + y - 2 = y + x - 2 = y \oplus x$
2. For all $x, y, z \in S$, $(x \oplus y) \oplus z = (x + y - 2) \oplus z = x + y + z - 4 = x \oplus (y + z - 2) = x \oplus (y \oplus z)$
3. Take $\vec{0} = 2 \in S$. Then for all x, $x \oplus \vec{0} = x + 2 - 2 = x$
4. For all $x \in S$, on $y = 4 - x \in S$, $x \oplus y = x + (4 - x) - 2 = 2 = \vec{0}$
5. For all $x \in S$, $1 \odot x = 1x + 2(1 - 1) = x$
6. For all $x \in S$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \odot (\beta \odot x) = \alpha \odot (\beta x + 2(1 - \beta)) = \alpha(\beta x + 2(1 - \beta)) + 2(1 - \alpha) = \alpha\beta x + 2(\alpha(1 - \beta) + (1 - \alpha)) = (\alpha\beta)x + 2(1 - \alpha\beta) = (\alpha\beta) \odot x$
7. For all $\alpha \in \mathbb{R}$ and $x, y \in S$, $\alpha \odot (x \oplus y) = \alpha(x + y - 2) + 2(1 - \alpha) = (\alpha x + 2(1 - \alpha)) + (\alpha y + 2(1 - \alpha)) - 2 = (\alpha \odot x) \oplus (\alpha \odot y)$
8. For all $\alpha, \beta \in \mathbb{R}$ and $x \in S$, $(\alpha + \beta) \odot x = (\alpha + \beta)x + 2(1 - \alpha - \beta) = (\alpha x + 2(1 - \alpha)) + (\beta x + 2(1 - \beta)) - 2 = (\alpha \odot x) \oplus (\beta \odot x)$

As all conditions are satisfied, S is a real vector space with the given operations.

As seen in the proof, its zero vector is 2.

2 Subspace

A subspace of a vector space is its subset which is also a vector space with the same linear structure. Particularly, this means that the subspace has the same scalar field, addition, scalar multiplication, and zero vector.

In the lecture the following theorem is shown:

Theorem 2.1. *If V is a vector space (over scalar field F) and $U \subseteq V$ (as a subset), then U is a subspace of V if and only if all of the following hold:*

- $0 \in U$

- *for all* $x, y \in U$, $x + y \in U$
- *for all* $x \in U$ *and* $\alpha \in F$, $\alpha x \in U$

It is sometimes tempting to show a set U to be a vector space by showing only the above 3 conditions. However, the above theorem works *only* when there is already a vector space V that contains U as a subset and has the same linear structure. For this purpose, typically

- either you choose V that is known to be a vector space (e.g. $\mathbb{R}^n$), which heavily limits its linear structure;
- or you construct one such V with the desired linear structure, which requires a proof that such V is a vector space, and is usually the same (if not harder) as working on U directly

2.1 Sum

See textbook Sec. 1.3 Q23–30.

You may already know from lecture that the intersection of two subspaces is still a subspace, but not necessarily their union, as taking union (usually) does not respect the linear structure.[2] So, is there an operation that plays a similar role?

The sum of two subsets S_1, S_2 of a vector space V is the set $S_1 + S_2 = \{\, x + y \mid x \in S_1,\ y \in S_2 \,\}$. Furthermore, if S_1, S_2 are both subspaces of V and satisfy the two conditions

- $S_1 \cap S_2 = \{0\}$
- $S_1 + S_2 = V$

Then we say the sum $S_1 + S_2$ is a direct sum and denote it as $V = S_1 \oplus S_2$. This is related to the concept of *linear independence* (which may have been covered in the lectures[3]).

2.2 Exercises

Sec 1.3 Q23 (a). Let W_1 and W_2 be subspaces of a vector space V. Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

Solution: We first show that $W_1 + W_2$ is a subspace of V.

To do so, we can use the theorem from the lecture. That is, we show that $0 \in W_1 + W_2$, $x + y \in W_1 + W_2$ for all $x, y \in W_1 + W_2$, and $\alpha x \in W_1 + W_2$ for all $\alpha \in F$ and $x \in W_1$.

- Since W_1, W_2 are subspaces of V, by the property of subspace $0 \in W_1$ and $0 \in W_2$, so $0 = 0 + 0 \in W_1 + W_2$
- Let $x, y \in W_1 + W_2$. By the definition of $W_1 + W_2$, there exists $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$ such that $x = x_1 + x_2$, $y = y_1 + y_2$.
  Then $x + y = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2)$. As W_1, W_2 are subspaces, $x_1 + y_1 \in W_1$ and $x_2 + y_2 \in W_2$, so $x + y = (x_1 + y_2) + (x_2 + y_2) \in W_1 + W_2$
- Let $x \in W_1 + W_2$, $\alpha \in F$. By the definition of $W_1 + W_2$, there exists $x_1 \in W_1$ and $x_2 \in W_2$ such that $x = x_1 + x_2$.
  As W_1 is a subspace, $\alpha x_1 \in W_1$ and $\alpha x_2 \in W_2$, so $\alpha x = \alpha(x_1 + x_2) = (\alpha x_1) + (\alpha x_2) \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V.

We now show that $W_1 + W_2$ contains both W_1 and W_2.

Let $x \in W_1$. Since W_2 is a subspace, $0 \in W_2$. So, $x = x + 0 \in W_1 + W_2$. As $x \in W_1$ is arbitrary, $W_1 \subseteq W_1 + W_2$.

[2] See HW question Sec 1.3 Q19: union of two subspaces is a subspace iff one contains the other.
[3] The relevant lecture note is already on the course webpage.

Similarly, let $x \in W_2$. Since W_1 is a subspace, $0 \in W_1$. So, $x = 0 + x \in W_1 + W_2$. As $x \in W_2$ is arbitrary, $W_2 \subseteq W_1 + W_2$.

So, $W_1 + W_2$ contains both W_1 and W_2.

Therefore, $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

2. Let V be a vector space. Determine if the following hold on all subspaces W, U_1, U_2 of V:
   (a) $W \cap (U_1 + U_2) = (W \cap U_1) + (W \cap U_2)$
   (b) if $U_1 \subseteq U_2$, $(U_1 + W) \cap U_2 = U_1 + (W \cap U_2)$

Solution:

(a) This does not hold in general.

A counter-example: $V = \mathbb{R}^2$ is the usual 2D plane (as a real vector space), $U_1 = \{ (x, 0) \in V \mid x \in \mathbb{R} \}$, $U_2 = \{ (0, y) \in V \mid y \in \mathbb{R} \}$, $W = \{ (x, x) \in V \mid x \in \mathbb{R} \}$. It is easy to verify that

- U_1, U_2, W are all subspaces of V
- $U_1 + U_2 = V$, $W \cap (U_1 + U_2) = W$
- $W \cap U_1 = W \cap U_2 = \{0\}$, $(W \cap U_1) + (W \cap U_2) = \{0\}$
- $W \neq \{0\}$

Note

Here the detail is omitted for brevity, but in homeworks and tests you should give appropriate justification on the above statements.

(b) Let $v \in (U_1 + W) \cap U_2$. Then there exists $u_1 \in U_1$, $w \in W$ such that $v = u_1 + w$ and $v \in U_2$.
As $w = v - u_1$ with $v \in U_2$ and $u_1 \in U_1 \subseteq U_2$, $w \in U_2$, so $w \in W \cap U_2$. This implies $v = u_1 + w \in U_1 + (W \cap U_2)$.
As v is arbitrary, $(U_1 + W) \cap U_2 \subseteq U_1 + (W \cap U_2)$.

Let $v \in U_1 + (W \cap U_2)$. Then there exists $u_1 \in U_1$ and $w \in W \cap U_2$ such that $v = u_1 + w$, and we have both $w \in W$ and $w \in U_2$. So $v = u_1 + w \in U_1 + W$.
As $u_1 \in U_1 \subseteq U_2$ and $w \in U_2$, we have $v = u_1 + w \in U_2$. Hence $v \in (U_1 + W) \cap U_2$.
As $v \in U_1 + (W \cap U_2)$ is arbitrary, $U_1 + (W \cap U_2) \subseteq (U_1 + W) \cap U_2$.

Therefore, $(U_1 + W) \cap U_2 = U_1 + (W \cap U_2)$.

Note

In the second part, you can also prove it with some comparison between sets.