Topic#14
Gram-Schmidt orthogonalization

Def. Let V be an i.p.s. with $\langle \cdot, \cdot \rangle$.

1°. x and y in V are **orthogonal if** $\langle x, y \rangle = 0$.

2°. A subset S of V is orthogonal **if** any two distinct vectors in S are orthogonal.

3°. $x \in V$ is called a unit vector **if** $\|x\| = 1$.

4°. A subset S of V is **orthonormal if** S is orthogonal and S consists entirely of unit vectors..

Note:

1°. Let $S = \{v_1, v_2, \cdots\}$ (can be infinite). Then S is orthonormal **iff**

$$\langle v_i, v_j \rangle = \delta_{ij}, i, j = 1, 2, \cdots$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Note: δ_{ij} is called the Kronecker delta.

2°. Let $0 \neq x \in V$. Then

$$\|x\| > 0, \text{ and } \frac{x}{\|x\|} \in V \text{ is a unit vector.}$$

Such process is called the **normalizing.**

e.g. Recall $H =$ set of complex-valued continuous f'ns on $[0, 2\pi]$.

$$\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt, f, g \in H.$$

H is an i.p.s with the I.p. $\langle\cdot,\cdot\rangle$.

$$S \overset{def}{=} \{e^{int} : n = 0, \pm 1, \pm 2, \cdots\}$$

We may show:
1°. $S \subset H$.
2°.

$$\begin{aligned}\langle e^{imt}, e^{int}\rangle &= \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\overline{e^{int}}dt = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)t}dt \\ &= \begin{cases}\text{if } m \neq n, \text{then} = \frac{1}{2\pi}\cdot\frac{1}{m-n}e^{i(m-n)t}\ |_0^{2\pi} = 0 \\ \text{if } m = n, \text{then} = \frac{1}{2\pi}\int_0^{2\pi} dt = 1\end{cases} \\ &= \delta_{mn}.\end{aligned}$$

$\therefore$ S is an orthonormal subset of H. □

Goal: Given an i.p.s. V, start from a linearly indep. subset S,

- to construct an orthogonal set S' such that

  1° span(S')=span(S)
  2° S' is l.indep.

- and further to normalize each vector in S' to get an orthonormal set S'' such that

  1° span(S'')=span(S')=span(S).
  2° S" is l.indep.

Theorem. Let V be an i.p.s. with $\langle\cdot,\cdot\rangle$.
Let $S=\{w_1,\cdots,w_n\}$ be a l. indep. subset of V.
Define $S'=\{v_1,\cdots,v_n\}$ as

$$\begin{cases} v_1 = w_1, \\ v_2 = w_2 - \frac{\langle w_2,v_1\rangle}{\|v_1\|^2}v_1, \\ v_3 = w_3 - \frac{\langle w_3,v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle w_3,v_2\rangle}{\|v_2\|^2}v_2, \\ \cdots \\ v_n = w_n - \frac{\langle w_n,v_1\rangle}{\|v_1\|^2}v_1 - \frac{\langle w_n,v_2\rangle}{\|v_2\|^2}v_2 - \cdots - \frac{\langle w_n,v_{n-1}\rangle}{\|v_{n-1}\|^2}v_{n-1}, \end{cases}$$

(called the G.-S. process) namely, for $k=2,\cdots,n$,

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j\rangle}{\|v_j\|^2} v_j.$$

Then, S' is an orthogonal set of nonzero vectors such that span$(S')=$span(S).

Pf. Induction on n:

$n=1$: $S=\{w_1\}$ l. Indep. $\therefore w_1 \neq 0$. $S'=\{v_1\}$, $v_1=w_1\neq 0$.
$\therefore$ TRUE

Assume "TRUE" for $n \geq 1$, to show "TRUE" for $n+1$:
Let $S=\{w_1,\cdots,w_{n+1}\}$ be l. indep. From $\{w_1,\cdots,w_n\}$, apply I.A. to get $\{v_1,\cdots,v_n\}$ which is an orthogonal set of nonzero vectors and span($\{v_1,\cdots,v_n\}$)=span($\{w_1,\cdots,w_n\}$).
Define $v_{n+1}=w_{n+1}-\sum_{j=1}^{n}\frac{\langle w_{n+1},v_j\rangle}{\|v_j\|^2}v_j$ (well-defined!).
1°. $v_{n+1}\neq 0$. Otherwise, $v_{n+1}=0$, implies $w_{n+1}\in$span($\{v_1,\cdots,v_n\}$)= span($\{w_1,\cdots,w_n\}$) which is a contradiction to the fact that $\{w_1,\cdots,w_{n+1}\}$ is l. indep.
2°. For $1\leq i\leq n$,

$$\begin{aligned}\langle v_{n+1},v_i\rangle &= \langle w_{n+1},v_i\rangle-\sum_{j=1}^{n}\frac{\langle w_{n+1},v_j\rangle}{\|v_j\|^2}\langle v_j,v_i\rangle\\ &= \langle w_{n+1},v_i\rangle-\frac{\langle w_{n+1},v_i\rangle}{\|v_i\|^2}\langle v_i,v_i\rangle \quad (\because v_1,\cdots,v_n \text{ is orthogoanl})\\ &= \langle w_{n+1},v_i\rangle-\langle w_{n+1},v_i\rangle=0.\end{aligned}$$

$\therefore S'=\{v_1,\cdots,v_{n+1}\}$ is an orthogonal set of nonzero vectors.

It remains to show: span(S')=span(S), namely

$$\text{span}(\{v_1, \cdots, v_{n+1}\}) = \text{span}(\{w_1, \cdots, w_{n+1}\}).$$

Claim 1. span(S')$\subset$ span(S)

Proof of Claim 1.

In fact, it suffices to show: $v_1, \cdots, v_{n+1} \in \text{span}(S)$

$v_1 = w_1 \in S$, $v_2 \in$span$(\{w_2, v_1\}) \subset \text{span}(\{w_1, w_2\})$

$v_3 \in$span$(\{w_3, v_1, v_2\}) \subset \text{span}(\{w_1, w_2, w_3\})$

$\vdots$ (Induction)

$v_{n+1} \in$span$(\{w_{n+1}, v_1, \cdots, v_n\}) \subset \text{span}(\{w_1, \cdots, w_{n+1}\})$

$\therefore$ span(S') is a subspace of span(S).

Claim 2. Any orthogonal subset of V consisting of nonzero vectors must be linearly independent.

Proof of Claim 2. Let $S \subset V$ be an orthogonal set of nonzero vectors. To show S is l. Indep.

Assume: $\sum_{i=1}^{m} a_i v_i = 0$ with $v_1, \cdots, v_m \in S$. We need to show that all a_i are zero. Indeed, for $1 \leq j \leq m$,

$$\begin{aligned} 0 &= \langle 0_v, v_j \rangle \\ &= \langle \sum_{i=1}^{m} a_i v_i, v_j \rangle \\ &= \sum_{i=1}^{m} a_i \langle v_i, v_j \rangle \\ &= a_j \langle v_j, v_j \rangle \\ &= a_j \|v_j\|^2. \end{aligned}$$

As $\|v_j\| > 0$, $a_j = 0$ $(j = 1, \cdots, m)$. □

For S is a l.indep. set, $\dim(\text{span}(S)) = \text{n+1}$ and S' is also l.indep. with $\sharp S' = n + 1$, so S' is a basis for span (S'). Then

$$\text{span}(S') = \text{span}(S).$$

□

Remark: Consider V with $\dim(V) = n$

$\beta = \{w_1, \cdots, w_n\}$: o.b. for V

↓ G.-S. process

$\beta' = \{v_1, \cdots, v_n\}$: orthogonal o.b. for V

↓ normalizing: $u_i := \frac{v_i}{\|v_i\|}$

$\beta'' = \{u_1, \cdots, u_n\}$: ortonormal o.b. for V

Example. $V = P_2(\mathbb{R}), \langle f, g\rangle \stackrel{def}{=} \int_{-1}^{1} f(t)g(t)dt$.
$\beta = \{1, x, x^2\}$: s.o.b.
Question: Complete the previous program to get β' & β''.

1°. Find $\beta' = \{v_1, v_2, v_3\}$ (orthogonal).

$v_1 = 1$

$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 : \|v_1\|^2 = \int_{-1}^{1} 1^2 dt = 2,$
$\langle w_2, v_1\rangle = \int_{-1}^{1} x \cdot 1 dx = 0$
$\therefore v_2 = x - \frac{0}{2} \cdot 1 = x$

$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$

$$\left.\begin{aligned} \langle w_3, v_1\rangle &= \int_{-1}^{1} x^2 \cdot 1 dx = \tfrac{2}{3} \\ \langle w_3, v_2\rangle &= \int_{-1}^{1} x^2 \cdot x dx = 0 \end{aligned}\right\} \therefore v_3 = x^2 - \frac{\frac{2}{3}}{2} \cdot 1 - 0 = x^2 - \frac{1}{3}$$

$\therefore \beta' = \{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal basis for $P_2(\mathbb{R})$.

2°. Find $\beta'' = \{u_1, u_2, u_3\}$ (orthonormal).

Note: $\|v_1\|^2 = 2$

$\|v_2\|^2 = \int_{-1}^{1} x^2 dx = \frac{2}{3}$

$\|v_3\|^2 = \int_{-1}^{2} (x^2 - \frac{1}{3})^2 dx = \frac{8}{45}$

$\therefore u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$

$u_2 = \frac{v_2}{\|v_2\|} = \frac{x}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}x$

$u_3 = \frac{v_3}{\|v_3\|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})$.

$$\therefore \beta'' = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})\}$$

is an orthonormal basis for $P_2(\mathbb{R})$. □

Remark:

$$\beta = \{1, x, x^2, \cdots\} \text{ s.o.b. for } P(\mathbb{R})$$
$$\downarrow \text{G.-S. process}$$
$$\beta' = \{1, x, x^2 - \tfrac{1}{3}, \cdots\} \text{ orthogonal basis for } P(\mathbb{R}).$$

Vectors in β' are called the Legendre polynomials.

The rest of this topic is to discuss:

Why orthogonal (orthonormal) set (basis) useful and important?

Theorem. Let V be an i.p.s. with $\langle \cdot, \cdot \rangle$. Let $S = \{v_1, \cdots, v_k\} \subset V$ be an orthogonal subset of nonzero vectors. Then for $x \in \text{span}(S)$,

$$x = \sum_{i=1}^{k} \frac{\langle x, v_i \rangle}{\|v_i\|^2} v_i.$$

Particularly, if S is further orthonormal, then

$$x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i.$$

Pf. Let $x \in$span(S) =span$(\{v_1, \cdots, v_k\})$, then

$$x = \sum_{i=1}^{k} a_i v_i \text{ for some } a_1, \cdots, a_k \in \mathbb{F}.$$

For $1 \leq j \leq k$,

$$\begin{aligned}
\langle x, v_j \rangle &= \langle \sum_{i=1}^{k} a_i v_i, v_j \rangle \\
&= \sum_{i=1}^{k} a_i \langle v_i, v_j \rangle \\
&= a_j \|v_j\|^2 \quad \text{(Because } S \text{ is orthogonal.)}
\end{aligned}$$

$\therefore a_j = \frac{\langle x, v_j \rangle}{\|v_j\|^2}, \quad j = 1, \cdots, k.$ □

Theorem. Let V be a nonzero i.p.s. with $\dim(V) < \infty$. Then,

(1) V has an orthonormal basis β.

(2) Let $\beta = \{v_1, \cdots, v_n\}$ is an orthonormal basis for V, then for each $v \in V$,

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$

Pf. Direct consequence of previous results. □

e.g.: Recall that for $P_2(\mathbb{R})$,

$\beta = \{w_1, w_2, w_3\} = \{1, x, x^2\}$: s.o.b.
$\overset{G.-S.}{\longrightarrow} \beta' = \{v_1, v_2, v_3\} = \{1, x, x^2 - \frac{1}{3}\}$: orthogonal basis
$\overset{Normalize}{\longrightarrow} \beta'' = \{u_1, u_2, u_3\} = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3})\}$:
orthonormal basis

Then, for any $f \in P_2(\mathbb{R})$,

$$f = \sum_{i=1}^{3} \langle f, u_i \rangle u_i.$$

For instance,
$f(x) = 1 + 2x + 3x^2 \in P_2(\mathbb{R})$,
$\langle f, u_1 \rangle = \int_{-1}^{1} (1 + 2x + 3x^2) \cdot \frac{1}{\sqrt{2}} dx = 2\sqrt{2}$
$\langle f, u_2 \rangle = \int_{-1}^{1} (1 + 2x + 3x^2) \cdot \sqrt{\frac{3}{2}} x dx = \frac{2\sqrt{6}}{3}$
$\langle f, u_3 \rangle = \int_{-1}^{1} (1 + 2x + 3x^2) \cdot \sqrt{\frac{45}{8}} (x^2 - \frac{1}{3}) dx = \frac{2\sqrt{10}}{5}$
$\therefore 1 + 2x + 3x^2 = f(x) = 2\sqrt{2} u_1 + \frac{2\sqrt{6}}{3} u_2 + \frac{2\sqrt{10}}{5} u_3.$ $\square$

An application:

> <u>**Prop.**</u> Let $T \in \mathcal{L}(V)$, where V is a finite-dim. i.p.s. with an orthonormal basis $\beta = \{v_1, \cdots, v_n\}$. Set $A = [T]_\beta$. Then,
>
> $$A_{ij} = \langle T(v_j), v_i \rangle, \quad 1 \leq i, j \leq n.$$

Pf. Since β is an orthonormal basis for V,

$$T(v_j) = \sum_{i=1}^{n} \langle T(v_j), v_i \rangle v_i.$$

Then, by def of $[T]_\beta$,

$$A_{ij} = \langle T(v_j), v_i \rangle.$$

□