

Topic#13

Inner product space

ALL YOU WILL SEE

V : v.s. over $\mathbb{F} = \mathbb{C}$ or \mathbb{R} , with “+” & “.”

Goal: Given a vector space V , introduce **an inner product** to discuss the length, distance, orthogonality of vectors in V .

↪ more geometric applications

Eg.1 Projection of a vector on a subspace.

Eg.2 $\beta = \{v_1, \dots, v_n\}$ is an o.b. for V . Through Gram-Schmidt process, we get $\beta' = \{v'_1, \dots, v'_n\}$ is an orthogonal o.b. for V .

Eg.3 $A \in M_{n \times n}(\mathbb{F})$ diagonalizable if and only if \exists an o.b. β for \mathbb{F}^n . consisting entirely of e-vectors of A . In what situation, can β be orthogonal?

$\mathbb{F} = \mathbb{C}$: A is a normal matrix ($AA^* = A^*A$).

$\mathbb{F} = \mathbb{R}$: A is a symmetric matrix ($A^T = A$).

Def. V : v.s. over \mathbb{F} ($=\mathbb{C}$ or \mathbb{R}). An **inner product** on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}, \quad (x, y) \mapsto \langle x, y \rangle$$

such that

- (a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle, \forall x, y, z \in V.$
- (b) $\langle cx, y \rangle = c\langle x, y \rangle, \forall x, y \in V, \forall c \in \mathbb{F}.$
- (c) $\overline{\langle x, y \rangle} = \langle y, x \rangle, \forall x, y \in V.$
- (d) If $x \neq 0_v$, then $\langle x, x \rangle$ is real and $\langle x, x \rangle > 0.$

Complex conjugate: $\overline{a + bi} = a - bi, a, b \in \mathbb{R}$

Note:

1°. $\langle \cdot, \cdot \rangle$ is linear in the first component (Exercise):

$$\left\langle \sum_{i=1}^m a_i x_i, y \right\rangle = \sum_{i=1}^m a_i \langle x_i, y \rangle.$$

2°. If $\mathbb{F} = \mathbb{R}$, then condition (c) reduces to $\langle x, y \rangle = \langle y, x \rangle$.

Quick properties of inner products:

Theorem. Let V be an i.p.s. with $\langle \cdot, \cdot \rangle$. Then,

- (1) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle, \forall x, y, z \in V.$
- (2) $\langle x, cy \rangle = \bar{c} \langle x, y \rangle, \forall x, y \in V, \forall c \in \mathbb{F}.$
- (3) $\langle x, 0_V \rangle = \langle 0_V, x \rangle = 0, \forall x \in V.$
- (4) $\langle x, x \rangle = 0$ iff $x = 0.$
- (5) If $\langle x, y \rangle = \langle x, z \rangle$ **for any** $x \in V$, then $y = z$. Particularly, if $\langle x, y \rangle = 0$ **for any** $x \in V$, then $y = 0_V.$

Proof:

$$(1): \overline{\langle x, y+z \rangle} \stackrel{(c)}{=} \langle y+z, x \rangle \stackrel{(a)}{=} \langle y, x \rangle + \langle z, x \rangle \stackrel{(c)}{=} \overline{\langle x, y \rangle} + \overline{\langle x, z \rangle} \\ = \overline{\langle x, y \rangle + \langle x, z \rangle}$$

□

$$(2): \overline{\langle x, cy \rangle} \stackrel{(c)}{=} \langle cy, x \rangle \stackrel{(b)}{=} c \langle y, x \rangle = \dots = \overline{\overline{c} \langle x, y \rangle}$$

□

$$(3): \langle x, 0_v \rangle = \langle x, 0 \cdot 0_v \rangle = \overline{0} \langle x, 0_v \rangle = 0$$

□

$$(4): \langle x, x \rangle = 0 \text{ iff } x = 0_v:$$

\Leftarrow obvious by (3)

\Rightarrow otherwise, $x \neq 0_v$, then by (d), $\langle x, x \rangle > 0$ contradiction
then $x = 0_v$

□

$$(5): \text{Easy proof in the simple situation: } \langle x, y \rangle = 0 \forall x \in V \Rightarrow y = 0$$

Generally, $\langle x, y \rangle = \langle x, z \rangle, \forall x \in V$, then

$0 = \langle x, y \rangle - \langle x, z \rangle = \langle x, y - z \rangle, \forall x \in V$. Apply the simple situation, $y - z = 0_v, \therefore y = z$.

□

Note:

1°. (1) & (2) mean:

$$\langle x, \sum_{i=1}^m b_i y_i \rangle = \sum_{i=1}^m \bar{b}_i \langle x, y_i \rangle$$

i.p. is conjugate linear in the 2nd component.

Therefore

$$\langle \sum_{i=1}^m a_i x_i, \sum_{i=1}^n b_i y_i \rangle = \sum_{i=1}^m \sum_{j=1}^n a_i \bar{b}_j \langle x_i, y_j \rangle.$$

2°. (5) is a useful characterization of zero vector (or two vectors equal to each other).

Def.: A v.s. V over \mathbb{F} equipped with an inner product is called an **inner product space**.

Remarks:

1°. V is a complex i.p.s. if $\mathbb{F} = \mathbb{C}$; V is a real i.p.s. if $\mathbb{F} = \mathbb{R}$.

2°. Let V be an i.p.s. with the i.p. $\langle \cdot, \cdot \rangle$.

Let W be a subspace of V .

Then W equipped with the same $\langle \cdot, \cdot \rangle$ is also an i.p.s.

3°. A v.s. V can be equipped with two different inner products and then this yields two different inner product spaces. For instance, for $V = P(\mathbb{R})$, we may let

$$\langle f, g \rangle_1 = \int_0^1 f(t)g(t)dt$$

or

$$\langle f, g \rangle_2 = \int_{-1}^1 f(t)g(t)dt.$$

These two are **distinct** inner products (why?).

Examples of inner products.

e.g. 1: $V = \mathbb{F}^n$. (scalar field = \mathbb{C} or \mathbb{R})

$$x = (x_1, \dots, x_n) \in \mathbb{F}^n$$

$$y = (y_1, \dots, y_n) \in \mathbb{F}^n$$

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum_{i=1}^n x_i \bar{y}_i.$$

Verify: $\langle \cdot, \cdot \rangle$ is an i.p. over \mathbb{F}^n . (Exercise)

Note:

1°. We call it the standard i.p. over \mathbb{F}^n .

2°. If $\mathbb{F} = \mathbb{R}$, then

$$\langle x, y \rangle \stackrel{\text{def}}{=} \sum_{i=1}^n x_i y_i \text{ (dot product } x \cdot y \text{ in } \mathbb{R}^n).$$

e.g. 2: $V = C([0, 1])$ (set of real-valued continuous f'ns on $[0, 1]$)
with $\mathbb{F} = \mathbb{R}$. Let

$$\langle \cdot, \cdot \rangle : (f, g) \in V \times V \mapsto \langle f, g \rangle \stackrel{\text{def}}{=} \int_0^1 f(t)g(t)dt \in \mathbb{R}.$$

Verify: $\langle \cdot, \cdot \rangle$ is an inner product on $V = C([0, 1])$.

- $\langle \cdot, \cdot \rangle$ is well-defined (obvious).
- Show (a), (b), (c), (d):

$$\left. \begin{array}{l} (a) \\ (b) \\ (c) \end{array} \right\} \text{obvious}$$

(d): Let $0 \neq f \in C([0, 1])$, then

$$\langle f, f \rangle = \int_0^1 f^2(t)dt > 0.$$

($\because f^2 > 0$ on some subinterval of $[0, 1]$)



e.g. 3:

An example to be used later: Let

$$H \stackrel{\text{def}}{=} \{\text{continuous complex-valued functions on } [0, 2\pi]\},$$

then H is a v.s. over \mathbb{F} . Let

$$\langle f, g \rangle \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{g(t)} dt, \quad f, g \in H.$$

Show: H with $\langle \cdot, \cdot \rangle$ as above is an i.p.s. ([Exercise](#))

Note: $f \in H \Leftrightarrow f = f_1 + if_2$ for some $f_1, f_2 \in C([0, 2\pi])$.

e.g. 4: $V = M_{n \times n}(\mathbb{F})$ ($\mathbb{F} = \mathbb{C}$ or \mathbb{R}). Let

$$\begin{aligned}\langle \cdot, \cdot \rangle: V \times V &\rightarrow \mathbb{F} \\ (A, B) &\mapsto \langle A, B \rangle \stackrel{\text{def}}{=} \text{tr}(B^* A).\end{aligned}$$

Verify: $\langle \cdot, \cdot \rangle$ is an i.p. on V .

Notation: For $A = [A_{ij}]_{n \times n} \in M_{n \times n}(\mathbb{F})$,

$$\begin{aligned}\mathbb{F} \ni \text{tr}(A) &\stackrel{\text{def}}{=} \sum_{i=1}^n A_{ii} : \text{trace of } A \\ M_{n \times n}(\mathbb{F}) \ni A^* &\stackrel{\text{def}}{=} \bar{A}^t, \text{ i.e. } (A^*)_{ij} = \bar{A}_{ji}: \text{conjugate transpose or} \\ &\text{adjoint of } A.\end{aligned}$$

Proof.

1°. $\langle \cdot, \cdot \rangle$ is well-defined. (obvious)

2°. Show (a), (b), (c), (d) in the definition:

Let $A, B, C \in M_{n \times n}(\mathbb{F})$, $c \in \mathbb{F}$.

(a)

$$\begin{aligned}\langle A + B, C \rangle &= \text{tr}(C^*(A + B)) = \text{tr}(C^*A + C^*B) \\ &= \text{tr}(C^*A) + \text{tr}(C^*B) = \langle A, C \rangle + \langle B, C \rangle.\end{aligned}$$

(b) $\langle cA, B \rangle = \text{tr}(B^*(cA)) = c \text{tr}(B^*A) = c \langle A, B \rangle.$

(c)

$$\begin{aligned}\overline{\langle A, B \rangle} &= \overline{\text{tr}(B^*A)} = \overline{\text{tr}(B^*A)^t} = \overline{\text{tr}(\bar{B}^t A)^t} = \overline{\text{tr}(A^t \bar{B})} \\ &= \text{tr}(\bar{A}^t B) = \text{tr}(A^* B) = \langle B, A \rangle.\end{aligned}$$

(d)

$$\begin{aligned}\langle A, A \rangle &= \text{tr}(A^*A) = \sum_{i=1}^n (A^*A)_{ii} = \sum_{i=1}^n \sum_{j=1}^n (A^*)_{ij} A_{ji} \\ &= \sum_{i=1}^n \sum_{j=1}^n \bar{A}_{ji} A_{ji} = \sum_{i=1}^n \sum_{j=1}^n |A_{ji}|^2.\end{aligned}$$

Therefore,

$$A \neq 0 \Rightarrow \langle A, A \rangle > 0.$$

(Equivalently, $\langle A, A \rangle = 0 \Rightarrow A = 0$ (zero matrix))

Note: $\langle A, B \rangle = \text{tr}(B^*A)$ called the Frobenius i.p.



Def. Let V be an i.p.s. with the i.p. $\langle \cdot, \cdot \rangle$. For $x \in V$,

$$\|x\| \stackrel{\text{def}}{=} \sqrt{\langle x, x \rangle}$$

is called the **norm** of $x \in V$, which is an i.p.s with $\langle \cdot, \cdot \rangle$.

e.g. $V = \mathbb{F}^n \ni x = (x_1, \dots, x_n)$:

$$\|x\| \stackrel{\text{def}}{=} \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^n x_i \bar{x}_i} = \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}.$$

\uparrow
Euclidean *length* of $x \in \mathbb{F}^n$.

Theorem. Let V be an i.p.s. over \mathbb{F} with $\langle \cdot, \cdot \rangle$. Then,

- (1) $\|x\| \geq 0, \forall x \in V$. And, $\|x\| = 0$ iff $x = 0_V$.
- (2) $\|cx\| = |c| \cdot \|x\|, \forall x \in V, \forall c \in \mathbb{F}$.
- (3) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$. (Triangle Inequality)

Pf.

Proof of (1) & (2): based on def of $\|\cdot\|$: $\|x\| = \sqrt{\langle x, x \rangle}$.

To show Thm(3), we need to show two lemmas.

Lemma1 (Pythagorean Thm):

If $\langle x, y \rangle = 0$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$

Proof: $\|x + y\|^2 \stackrel{\text{def}}{=} \langle x + y, x + y \rangle$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle = \langle x, x \rangle + \langle y, y \rangle = \|x\|^2 + \|y\|^2$$

□

Lemma2 (Cauchy-Schwarz inequality)

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \forall x, y \in V.$$

Proof: Let $x, y \in V$. True if $x = 0_V$ or $y = 0_V$. Assume: $x \neq 0_V$ and $y \neq 0_V$.

$$\text{Then take } z \stackrel{\text{def}}{=} y - \frac{\langle y, x \rangle}{\langle x, x \rangle} x,$$

$$\text{then } \langle z, x \rangle = \langle y - \frac{\langle y, x \rangle}{\langle x, x \rangle} x, x \rangle = \langle y, x \rangle - \frac{\langle y, x \rangle}{\langle x, x \rangle} \langle x, x \rangle = 0$$

Apply lemma 1 above to $y = z + \frac{\langle y, x \rangle}{\langle x, x \rangle} x$:

$$\|y\|^2 = \|z + \frac{\langle y, x \rangle}{\langle x, x \rangle} x\|^2 \stackrel{\text{lemma1}}{=} \|z\|^2 + \|\frac{\langle y, x \rangle}{\langle x, x \rangle} x\|^2 = \|z\|^2 + \frac{|\langle y, x \rangle|^2}{\|x\|^2}$$

$$\therefore \|y\|^2 \geq \frac{|\langle y, x \rangle|^2}{\|x\|^2}$$

$$\therefore |\langle x, y \rangle|^2 \leq \|x\| \|y\|.$$



Now we can show Thm(3), the triangle inequality.

$$\begin{aligned}\|x + y\|^2 &= \langle x + y, x + y \rangle \\ &= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle \\ &= \langle x, x \rangle + (\langle x, y \rangle + \overline{\langle x, y \rangle}) + \langle y, y \rangle \\ &= \|x\|^2 + 2\operatorname{Re}\langle x, y \rangle + \|y\|^2 \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \quad (-|z| \leq \operatorname{Re} z \leq |z|) \\ &\leq \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 \quad (\text{c.-s.}) \\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

So, $\|x + y\| \leq \|x\| + \|y\|$.

In fact, we can also construct a quadratic equation

$$0 \leq f(t) \stackrel{\text{def}}{=} \|x + ty\|^2 = \cdots = a + bt + ct^2 \text{ where } c \neq 0$$

$\therefore b^2 - 4ac \leq 0$. It implies C.S. inequality.

Examples for applications of C.S. Inequality & Triangle Inequality

- $V = \mathbb{F}^n$ with the s.i.p.:

$$\left| \sum_{i=1}^n x_i y_i \right| \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2}.$$

$$\left(\sum_{i=1}^n |x_i + y_i|^2 \right)^{1/2} \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} + \left(\sum_{i=1}^n |y_i|^2 \right)^{1/2}$$

- $V = C([a, b])$ with $\langle f, g \rangle = \int_a^b f(t)g(t)dt$:

$$\left| \int_a^b f(t)g(t)dt \right| \leq \left(\int_a^b |f(t)|^2 dt \right)^{1/2} \left(\int_a^b |g(t)|^2 dt \right)^{1/2},$$

$$\left(\left| \int_a^b |f(t) + g(t)|^2 dt \right| \right)^{1/2} \leq \left(\int_a^b |f(t)|^2 dt \right)^{1/2} + \left(\int_a^b |g(t)|^2 dt \right)^{1/2}.$$

