Topic#9
Change of coordinates

Let V: v.s., $\dim(V) = n < \infty$, β, β': o.b. for V.

$$
\begin{array}{ccc}
v \in V & \xrightarrow{I_V \in \mathcal{L}(V),\, invertible} & v \in V \\
\Big\downarrow [\cdot]_{\beta'} & & \Big\downarrow [\cdot]_{\beta} \\
[v]_{\beta'} \in \mathbb{F}^n & \xrightarrow{Q \overset{def}{=} [I_V]_{\beta'}^{\beta} \in M_{n\times n}(\mathbb{F})} & [v]_{\beta} \in \mathbb{F}^n
\end{array}
$$

Then,

$$[v]_\beta = [I_v]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'} \text{ where } Q \overset{def}{=} [I_v]_{\beta'}^{\beta}$$

$Q = [I_v]_{\beta'}^{\beta}$ is called a change of coordinate matrix that changes β'-coordinate of v to β-coordinate of v.

Notes:

1°. $Q^{-1} = ([I_v]_{\beta'}^{\beta})^{-1} = [I_v{}^{-1}]_{\beta}^{\beta'} = [I_v]_{\beta}^{\beta'}$: the change of coordinate matrix that changes β-coordinate of v to β'-coordinate of v.

2°. $\beta' = \{v_1', \cdots, v_n'\}$, $\beta = \{v_1, \cdots, v_n\}$

$Q = [I_v]_{\beta'}^{\beta} = ([I_v(v_1')]_\beta, \cdots, [I_v(v_n')]_\beta) = ([v_1']_\beta, \cdots, [v_n']_\beta)$

Prop: $V \overset{T \in \mathcal{L}(V)}{\longrightarrow} V$ where dim(V)$= n < \infty$ and β, β' : o.b. for V.

Relation of $[T]_\beta$ and $[T]_{\beta'}$?

$[T]_{\beta'} = [T]_{\beta'}^{\beta'} = [I_v \circ T \circ I_v]_{\beta'}^{\beta'} = [I_v]_{\beta}^{\beta'} \cdot [T]_{\beta}^{\beta} \cdot [I_v]_{\beta'}^{\beta} = Q^{-1}[T]_{\beta}^{\beta} Q$

Then, $[T]_{\beta'} = Q^{-1}[T]_\beta Q$, where $Q = [I_v]_{\beta'}^{\beta}$.

Remark:

$A \in M_{n\times n}(\mathbb{F})$

1°. Let $\gamma' = \{e_1, \cdots, e_n\}$ be s.o.b. for $\mathbb{F}^n$.

$[L_A]_{\gamma'} = ([L_A(e_1)]_{\gamma'}, \cdots, [L_A(e_n)]_{\gamma'}) = ([Ae_1]_{\gamma'}, \cdots, [Ae_n]_{\gamma'}) = A.$

Ae_1 : 1^{st} column of A, γ': s.o.b $\Rightarrow [Ae_1]_{\gamma'}$ is still 1^{st} column of A.

2° Let $\gamma = \{v_1, \cdots, v_n\}$ be an o.b. for $\mathbb{F}^n$.

$$[L_A]_\gamma = [L_A]_\gamma^\gamma = [I_{\mathbb{F}^n} \circ L_A \circ I_{\mathbb{F}^n}]_\gamma^\gamma = [I_{\mathbb{F}^n}]_{\gamma'}^\gamma [L_A]_{\gamma'}^{\gamma'} [I_{\mathbb{F}^n}]_\gamma^{\gamma'} = Q^{-1}AQ$$

$$\gamma'\text{: s.o.b. for } \mathbb{F}^n$$

And $Q \overset{def}{=} [I_{\mathbb{F}^n}]_\gamma^{\gamma'} = ([v_1]_{\gamma'} \mid [v_2]_{\gamma'} \mid \cdots \mid [v_n]_{\gamma'}) = (v_1 \mid v_2 \mid \cdots \mid v_n)$

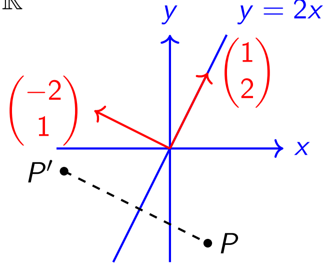
i.e. $Q = (v_1 \mid v_2 \mid \cdots \mid v_n)$

$\therefore [L_A]_{\gamma=\{v_1,\cdots,v_n\}} = Q^{-1}AQ$

where $Q = [I_v]_\gamma^{\gamma'} = (v_1 \mid v_2 \mid \cdots \mid v_n)_{n\times n}$.

e.g.

$\mathbb{R}^2$



$T \stackrel{def}{=}$ the reflection about the line $y = 2x$

Question: Find $[T]_\beta$, where β is the s.o.b. for $\mathbb{R}^2$.

Sln: $\beta' = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ is another o.b. for $\mathbb{R}^2$. By def. of T,

$$T \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 1 \end{pmatrix},$$

$$T \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-1) \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

$$\therefore [T]_{\beta'} = [T]_{\beta'}^{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $Q = [I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ (change β'-co to β-co).

$\therefore Q^{-1} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ (change β-co to β'-co).

Therefore,

$$\begin{aligned} [T]_{\beta} &= [I_{\mathbb{R}^2} \circ T \circ I_{\mathbb{R}^2}]_{\beta}^{\beta} \\ &= [I_{\mathbb{R}^2}]_{\beta'}^{\beta} [T]_{\beta'}^{\beta'} [I_{\mathbb{R}^2}]_{\beta}^{\beta'} \\ &= Q[T]_{\beta'} Q^{-1} \\ &= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} := A. \end{aligned}$$

Since β is a s.o.b. for $\mathbb{R}^2$, we have $T = L_A$. □