THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 3030 Abstract Algebra 2024-25
Homework 3
Due Date: 3rd October 2024

Compulsory Part

1. Let N be a normal subgroup of a group G, and let $m = [G : N]$. Show that $a^m \in N$ for every $a \in G$.

2. Prove that the **torsion subgroup** T (i.e. the set of all elements having finite orders) of an abelian group G is a normal subgroup of G, and that G/T is **torsion free** (meaning that the identity is the only element of finite order).

3. Let H and K be groups and let $G = H \times K$. Recall that both H and K appear as subgroups of G in a natural way. Show that these subgroups H (actually $H \times \{e\}$) and K (actually $\{e\} \times K$) have the following properties.

   (a) Every element of G is of the form hk for some $h \in H$ and $k \in K$.

   (b) $hk = kh$ for all $h \in H$ and $k \in K$.

   (c) $H \cap K = \{e\}$.

4. Let H and K be subgroups of a group G satisfying the three properties listed in the preceding exercise. Show that for each $g \in G$, the expression $g = hk$ for $h \in H$ and $k \in K$ is unique. Then let each g be renamed (h, k). Show that, under this renaming, G becomes structurally identical (isomorphic) to $H \times K$.

5. Let G, H, and K be finitely generated abelian groups. Show that if $G \times K$ is isomorphic to $H \times K$, then $G \simeq H$.

6. Suppose that H and K are normal subgroups of a group G with $H \cap K = \{e\}$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Optional Part

1. Given any subset S of a group G, show that it makes sense to speak of the smallest normal subgroup that contains S.

2. Prove that if a finite abelian group has order a power of a prime p, then the order of every element in the group is a power of p. Can the hypothesis of commutativity be dropped? Why, or why not?

3. Let G be a finite abelian group and let p be a prime dividing $|G|$. Prove that G contains an element of order p.

4. Show that a finite abelian group is not cyclic if and only if it contains a subgroup isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p.

5. Let G and G' be groups, and let N and N' be normal subgroups of G and G' respectively. Let ϕ be a homomorphism of G into G'. Show that ϕ induces a natural homomorphism $\phi_* : G/N \to G'/N'$ if $\phi(N) \subseteq N'$. (This fact is used constantly in algebraic topology.)

6. If a group N can be realized as a normal subgroup of two groups G_1 and G_2, and if $G_1/N \cong G_2/N$, does it imply that $G_1 \cong G_2$? Give a proof or a counterexample.

7. Suppose N is a normal subgroup of a group G such that N and G/N are finitely generated. Show that G is also finitely generated.

8. Suppose N is a normal subgroup of a group G which is cyclic. Show that every subgroup of N is normal in G.

9. Show that the isomorphism class of a direct product is independent of the ordering of the factors, i.e. $G_1 \times G_2 \times \cdots \times G_n$ is isomorphic to $G_{\sigma(1)} \times G_{\sigma(2)} \times \cdots \times G_{\sigma(n)}$ for any permutation $\sigma \in S_n$.