MATH5021 THEORY OF PDE I
Homework 4

Due Nov 21, 2024, in hard copies. Late assignment is not accepted.

1. [10 points] Consider the initial value problem:

$$\Box\phi = (\partial_{x_1}\phi)^3, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3,$$
$$\phi(0,\cdot) = \epsilon\phi_0, \quad \partial_t\phi(t,\cdot) = \epsilon\phi_1.$$

Assume $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^3)^2$, show that this IVP admits a solution in $H^3(\mathbb{R}^3)$ up to times of order ϵ^{-2}, whenever $\epsilon > 0$ is chosen sufficiently small.

2. [10 points] Denoting by $u = (u_1, u_2, u_3)$ the velocity and by p the pressure of an incompressible fluid, whose motion is described by the Euler equations:

$$\partial_t u + u \cdot \nabla u + \nabla p = 0, \quad \text{div } u = 0, \quad \text{in } [0,\infty) \times \mathbb{R}^3, \tag{1}$$
$$u(0,\cdot) = u_0. \tag{2}$$

   i. [5 points] Let $E(t) = \frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R}^3)}$. Show that $E(t)$ is a conserved quantity. (Hint: Test the equation with u and then integrate it in space.)

   ii. [5 points] Suppose the fluid is confined in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary, and u verifies the slip boundary condition

$$u \cdot N = 0, \quad \text{on } \partial\Omega,$$

   where N is the outward unit normal of $\partial\Omega$. Let $\mathcal{E}(t) = \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)}$. Show that $\mathcal{E}(t)$ is a conserved quantity.

3. [5 points] Consider the motion of a fluid described by the 3D Euler equations (1) with velocity u. Let $x(t) = (x_1(t), x_2(t), x_3(t))$ be the path followed by a fluid particle, i.e., the fluid's velocity u satisfies

$$\frac{dx(t)}{dt} = u(t, x(t)).$$

Show that

$$\frac{d^2x(t)}{dt^2} = -\nabla p.$$

This is the acceleration of the fluid particles.