MATH5021 THEORY OF PDE I
Homework 2

Due Oct 23, 2024, in hard copies. Late assignment is not accepted.

1. (8 points) Let $\phi = \phi(t,x)$ be the solution to the linear wave equation

$$\Box\phi = 0, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1.$$

   For fixed $R > 0$ and $x_0 \in \mathbb{R}^n$, we define

$$\mathcal{E}(t) = \frac{1}{2}\int_{B_{R-t}(x_0)} |\partial_{t,x}\phi(t,x)|^2\,dx,$$

   where we denote $B_{R-t}(x_0)$ by the ball centered at x_0 with radius $R-t$. Prove that $\mathcal{E}(t) \le \mathcal{E}(0)$.

2. (7 points) Let f be C^2-function defined on $\mathbb{R}^n$ equipped with a Riemannian metric g. For any vector fields X, Y, we define $\nabla^2 f$ (i.e., the Hessian) to be the bilinear form satisfying

$$\nabla^2 f(X,Y) = g(\nabla_X \nabla f, Y).$$

   Prove that

$$\nabla^2 f(X,Y) = XYf - (\nabla_X Y)f.$$

   Use this to conclude that $\nabla^2 f(X,Y) = \nabla^2 f(Y,X)$.

3. [10 points] Consider the initial value problem:

$$\begin{aligned} &\Box\phi = f(t,x,\phi), \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ &\phi(0,\cdot) = \phi_0, \quad \partial_t\phi(t,\cdot) = \phi_1. \end{aligned} \tag{1}$$

   Do the following:

   a. [8] Let $f(t,x,\phi) = |\phi|^2$. Given $(\phi_0,\phi_1) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, prove that (1) admits a local-in-time solution $\phi(t,\cdot) \in H^2(\mathbb{R}^3)$.
   (Hint: It is enough to prove the a priori energy estimate. The following Sobolev embedding may come in handy: $\|\phi\|_{L^6(\mathbb{R}^3)} \le C\|\phi\|_{\dot{H}^1(\mathbb{R}^3)}$.).

   b. [2] Let $f(t,x,\phi) = |\nabla\phi|^2$. Is it still possible to find a local-in-time solution when $(\phi_0,\phi_1) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$? Explain your answer. No rigorous justification is required.