Curves in $\mathbb{R}^n$

Defn: Let $I \subset \mathbb{R}$ be an interval. A (continuous) curve in $\mathbb{R}^n$ is a continuous (vector-valued) function

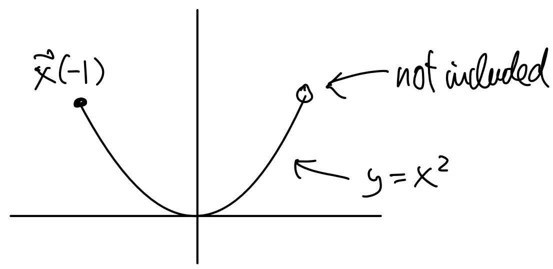
$$\vec{X} : I \to \mathbb{R}^n$$

i.e. $t \in I$, $\vec{X}(t) = (X_1(t), \cdots, X_n(t)) \in \mathbb{R}^n$ such that every component function $X_i(t)$ is continuous $(i = 1, \cdots, n)$

egs (i) $\vec{X} : [-1, 1) \to \mathbb{R}^2$

$$\vec{X}(t) = (t, t^2)$$

$$[\, x = t,\ y = t^2 \Rightarrow y = x^2 \,]$$



(ii) Of course, parametric form of a line gives a curve:

$$\vec{X}(t) = \vec{a} + t\vec{V}, \quad t \in (-\infty, \infty)$$

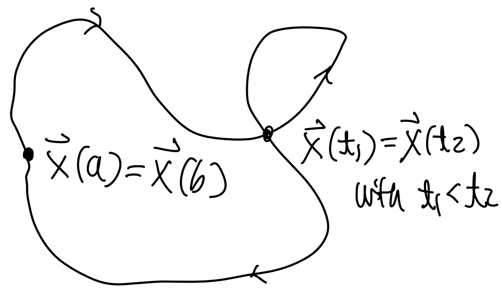
$$= (a_1 + tV_1, \ldots, a_n + tV_n)$$

Defn: A curve $\vec{X} : [a,b] \to \mathbb{R}^n$ is said to be

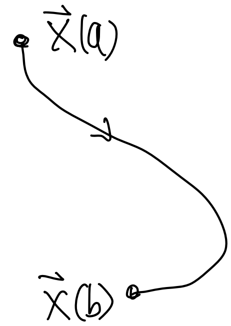
(i) closed if $\vec{X}(a) = \vec{X}(b)$

(ii) simple if $\vec{X}(t_1) \neq \vec{X}(t_2)$ for $a \le t_1 < t_2 \le b$ except possibly at $t_1 = a$ & $t_2 = b$.

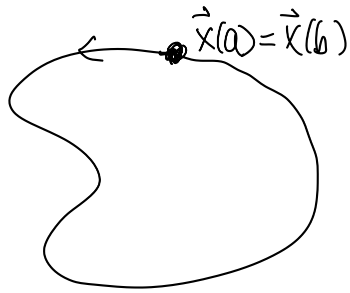
eg:



<u>closed</u>, <u>not simple</u>



<u>not closed</u>, <u>simple</u>



<u>Closed & simple</u>

(simple closed curve)

<u>Thm</u>: Let $\vec{x}(t)=(x_1(t),\cdots,x_n(t))$. Then

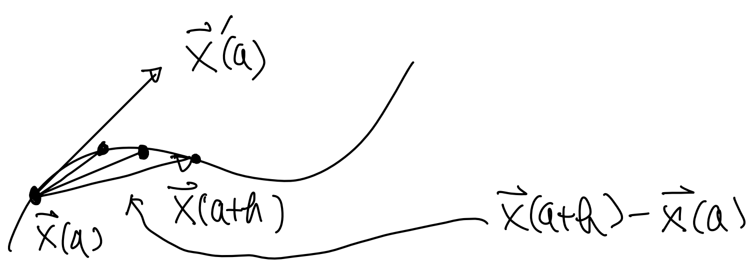
(1) $\lim\limits_{t\to a}\vec{x}(t)=\left(\lim\limits_{t\to a}x_1(t),\cdots,\lim\limits_{t\to a}x_n(t)\right)$

(2) $\vec{x}'(t)\overset{\text{def}}{=}\lim\limits_{h\to 0}\dfrac{\vec{x}(t+h)-\vec{x}(t)}{h}=(x_1'(t),\cdots,x_n'(t))$

(provided limits exist)

<u>Defn</u>: $\vec{x}'(a)$ = <u>tangent vector</u> of $\vec{x}(t)$ at $t=a$.

<u>Picture</u>:



<u>Physics</u>: If $\vec{x}(t)$ = <u>displacement</u> at time t

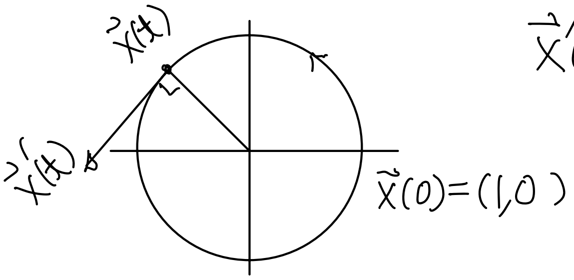
Then $\vec{x}'(t)$ = <u>velocity (vector)</u> at time t

$\vec{x}''(t)$ = <u>acceleration (vector)</u>

$\|\vec{x}'(t)\|$ = speed.

<u>eg</u>: $\vec{x}(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$

($\overset{\shortparallel}{x}$ $\overset{\shortparallel}{y}$ $\Rightarrow x^2+y^2=1$ the <u>unit circle</u>)



(simple closed)

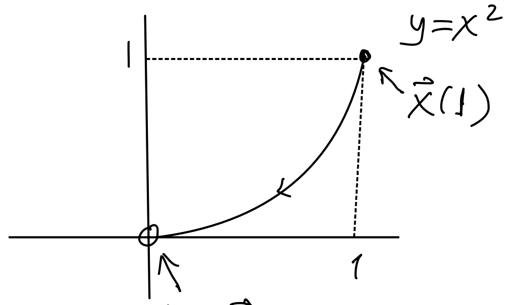
$\vec{x}'(t) = (-\sin t, \cos t)$ is the tangent vector

$$\left[\begin{array}{l} \vec{v} = \text{velocity} = \vec{x}'(t) = (-\sin t, \cos t) \\ \vec{a} = \text{acceleration} = \vec{x}''(t) = (-\cos t, -\sin t) \\ \qquad\qquad\qquad\qquad\qquad = -\vec{x}(t) \\ \text{speed} = \|\vec{x}'(t)\| = 1 \end{array}\right]$$

<u>eg</u> $\vec{x}: [1, \infty) \to \mathbb{R}^2$

$\vec{x}(t) = \left(\frac{1}{t}, \frac{1}{t^2}\right)$

($\overset{\shortparallel}{x}$ $\overset{\shortparallel}{y}$ $\Rightarrow y = x^2$)



$\lim_{t\to\infty} \vec{x}(t) = (0,0)$

(not included in the curve)

Rules

Let $\vec{x}(t), \vec{y}(t)$ be curves in $\mathbb{R}^n$, $c \in \mathbb{R}$ be a constant

$f(t)$ be a real-valued function. Then

(1) $\left(\vec{x}(t)+\vec{y}(t)\right)' = \vec{x}'(t)+\vec{y}'(t)$

(2) $\left(c\,\vec{x}(t)\right)' = c\,\vec{x}'(t)$

(3) $\left(f(t)\,\vec{x}(t)\right)' = f'(t)\,\vec{x}(t) + f(t)\,\vec{x}'(t)$

(4) $\left(\vec{x}(t)\cdot\vec{y}(t)\right)' = \vec{x}'(t)\cdot\vec{y}(t) + \vec{x}(t)\cdot\vec{y}'(t)$

(5) For $n=3$,

$$\left(\vec{x}(t)\times\vec{y}(t)\right)' = \vec{x}'(t)\times\vec{y}(t) + \vec{x}(t)\times\vec{y}'(t)$$

<u>Remark</u>: (3), (4) & (5) are all called <u>product rules</u>.

Arclength (of a curve)

Let $\vec{x}(t)$ be a curve with $\vec{x}'(t)$ exists and continuous

<u>Def</u>: <u>Arclength</u> of $\vec{x}(t)$ for $a \leq t \leq b$ is

$$s = \int_a^b \|\vec{x}'(t)\|\,dt$$

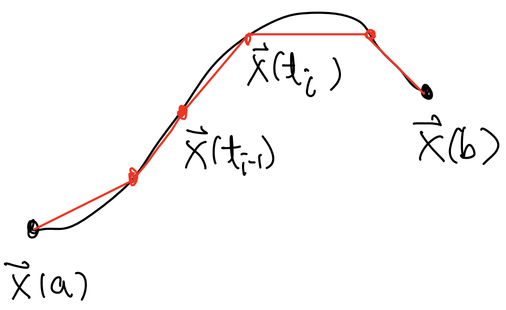
<u>Remark</u>: If $\vec{x}(t)$ = displacement at time t

then $\vec{x}'(t)$ = velocity

$\|\vec{x}'(t)\|$ = speed

$\int_a^b \|\vec{x}'(t)\|\,dt$ = distance travelled.

Idea of the defn (from mathematician's point of view)



approximate the curve by straight line segments.

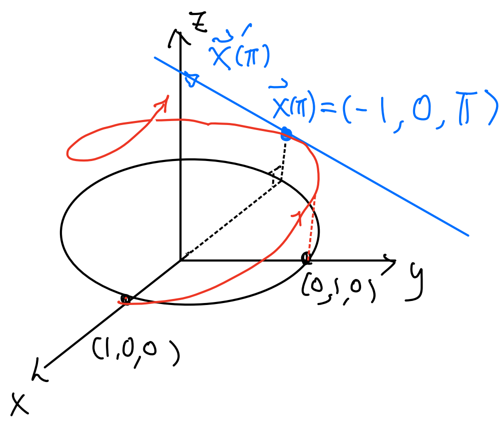
$$s \approx \sum_i \|\vec{x}(t_i) - \vec{x}(t_{i-1})\| \qquad \left(\vec{x}'(t_i) = \lim_{t \to t_i} \frac{\vec{x}(t) - \vec{x}(t_i)}{t - t_i} \right)$$

$$\approx \sum_i \|\vec{x}'(t_i)\| (t_i - t_{i-1}) \longrightarrow \int_a^b \|\vec{x}'(t)\| \, dt$$

"as the approximation get better & better"

<u>eg</u>: (Helix)

$\vec{x}(t) = (\cos t, \sin t, t)$ (curve in $\mathbb{R}^3$) $t \in [0, 2\pi]$



(a) Find the tangent line of $\vec{x}$ at $t = \pi$

(b) Find arclength of the helix

<u>Soln</u> (a): $\vec{x}(t) = (\cos t, \sin t, t)$

$\Rightarrow \vec{x}'(t) = (-\sin t, \cos t, 1)$

$\therefore \vec{x}'(\pi) = (0, -1, 1)$ is the tangent vector of $\vec{x}$ at $t = \pi$

Clearly $\vec{X}(\pi) = (-1, 0, \pi)$ is a point on the tangent line

$\therefore$ The tangent line at $t = \pi$ is given by

$$\vec{Y}(\tau) = \vec{X}(\pi) + \tau \vec{X}'(\pi) \qquad \tau \in (-\infty, \infty)$$

$$= (-1, 0, \pi) + \tau(0, -1, 1)$$

(b) $\quad \|\vec{X}'(t)\| = \sqrt{(-\sin t)^2 + \cos^2 t + 1^2} = \sqrt{2}$

$\Rightarrow$ arc-length $\quad s = \int_0^{2\pi} \|\vec{X}'(t)\| \, dt = \int_0^{2\pi} \sqrt{2} \, dt$

$$= 2\sqrt{2}\pi \quad ※$$

<u>Remark</u>: Arclength is independent of change of parameters! (Proof omitted) (Ex)