Chap 8. Limiting thms.

§8.1 Introduction

Q: Let $X_1, X_2, \cdots,$ be a sequence of independent, identically distributed r.v.'s. What can we say about the limiting behavior of

$$\frac{X_1 + \cdots + X_n}{n} \quad \text{as} \quad n \to \infty \ ?$$

§ 8.2. Two basic inequalities

Prop 8. (Markov inequality)

Let X be a non-negative r.v. Then for any $a > 0$,

$$P\{X \geq a\} \leq E[X]/a.$$

Pf. Let $I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{otherwise.} \end{cases}$

Then I is a r.v. so that $I \leq \frac{X}{a}$

(here we use the fact $X \geq 0$)

So $E[I] \leq E[\frac{X}{a}]$

But $E[I] = 1 \cdot P\{I=1\} + 0 \cdot P\{I=0\}$

$= P\{X \geq a\}$.

Hence $P\{X \geq a\} \leq E[X]/a$.

Prop 9. (Chebyshev's inequality)

Let X be a r.v. with finite mean μ and variance σ^2. Then for any $\varepsilon > 0$,

$$P\{|X-\mu| \geq \varepsilon\} \leq \sigma^2/\varepsilon^2$$

Pf. Let $Y = |X-\mu|^2$. Applying Markov inequality to Y gives

$$P\{|X-\mu| \geq \varepsilon\} = P\{Y \geq \varepsilon^2\} \leq \frac{E[Y]}{\varepsilon^2}$$

$$\leq \quad E[(X-\mu)^2]/\varepsilon^2$$

$$= \quad Var(X)/\varepsilon^2 = \sigma^2/\varepsilon^2.$$

Example.

Suppose that it is known that the number of items produced in a factory during a week is a random variable with mean 50.

(a) What can be said about the probability that this week's production will exceed 75?

(b) If the variance of a week's production is known to equal 25, then what can be said about the probability that this week's production will be between 40 and 60?

Solution: Let X be the number of items produced in a week. $E[X] = 50$

Then

(a) By Markov, $P\{X > 75\} \leq \dfrac{E[X]}{75} = \dfrac{2}{3}$.

(b) Since $\sigma^2 = 25$,

$$P\{40 < X \leq 60\} = P\{|X-50| \leq 10\}$$

$$\geq 1 - \frac{\sigma^2}{10^2}$$

$$\geq 1 - \frac{25}{100} = \frac{3}{4} = 0.75.$$

$\boxed{}$.

Prop 1. Let X be a r.v. with a finite mean μ. Suppose $Var(X)=0$. Then

$$P\{X=\mu\}=1.$$

Proof. $P\{X\neq\mu\} = P\{\bigcup_{k=1}^{\infty} |X-\mu| \geq \frac{1}{k}\}$

(countable sub-additivity)

$$\leq \sum_{k=1}^{\infty} P\{|X-\mu| \geq \frac{1}{k}\}$$

(Chebyshev)

$$\leq \sum_{k=1}^{\infty} \frac{Var(X)}{(\frac{1}{k})^2} = 0$$

Hence $P\{X\neq\mu\}=0$

So

$$P\{X=\mu\} = 1 - P\{X \neq \mu\} = 1.$$

$\blacksquare$

Thm 2 (The weak law of large numbers)

Let $X_1, X_2, \cdots, X_n, \cdots$ be an i.i.d sequence of r.v's, having a finite mean. Then for any $\varepsilon > 0$,

$$P\left\{ \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| \geq \varepsilon \right\} \to 0 \text{ as } n \to \infty.$$

Pf. We prove the thm under an additional assumption that $Var(X_i) =: \sigma^2 < \infty$.

$$E\left[\frac{X_1 + \cdots + X_n}{n}\right] = \frac{1}{n} \sum_{k=1}^{n} E[X_k] = \mu.$$

$$Var\left(\frac{X_1 + \cdots + X_n}{n}\right) = \frac{1}{n^2} Var(X_1 + \cdots + X_n)$$

$$= \frac{1}{n^2} \cdot \sum_{k=1}^{n} Var(X_k) \quad (\text{since } X_1, \cdots, X_n \text{ are independent})$$

$$= \frac{\sigma^2 \cdot n}{n^2} = \frac{\sigma^2}{n}.$$

Applying the Chebyshev inequality to $\dfrac{X_1+\cdots+X_n}{n}$,

we obtain

$$P\left\{ \left| \frac{X_1+\cdots+X_n}{n} - \mu \right| \geq \varepsilon \right\} \leq \frac{\mathrm{Var}\left(\frac{X_1+\cdots+X_n}{n}\right)}{\varepsilon^2}$$

$$= \frac{\sigma^2}{n\varepsilon^2}$$

$$\to 0 \quad \text{as } n \to \infty.$$

$\square$

Thm 3 (The central limit Thm).

Let $X_1, \cdots, X_n, \cdots,$ be an i.i.d. sequence of r.v.'s, each having a finite mean μ and a finite variance σ^2.

Then $\forall\, a \in \mathbb{R}$,

$$P\left\{ \frac{X_1+\cdots+X_n - n\mu}{\sqrt{n}\,\sigma} \leq a \right\} \longrightarrow \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{x^2}{2}}\,dx$$

as $n \to \infty$.

That is, the distribution of

$$\frac{X_1+\cdots+X_n - n\mu}{\sqrt{n}\,\sigma} = \frac{X_1+\cdots+X_n - n\mu}{\sqrt{\mathrm{Var}(X_1+\cdots+X_n)}}$$

converges to the standard normal distribution, as $n \to \infty$.

The proof of the above Thm will be given in the next class.

Example 4. If 10 fair dice are rolled, find the approximate prob. that the sum obtained is between 30 and 40.

Solution: Let X_i be the value obtained in the i-th roll, $i = 1, 2, \cdots, 10$.

We need to calculate

$$P\{\underset{\text{Continuity correction}}{29.5} \leq X_1 + \cdots + X_{10} \leq \underset{\text{Continuity correction}}{40.5}\}$$

Notice that $\mu = E[X_i] = \frac{1}{6}(1 + 2 + \cdots + 6)$

$= 7/2$

$$E[X_i^2] = \frac{1}{6}(1^2 + 2^2 + \cdots + 6^2)$$

$$= \frac{1}{6} \cdot \frac{6 \times 7 \times 13}{6}$$

$$\sigma^2 = Var(X_i) = E[X_i^2] - E[X_i]^2$$

$$= \frac{35}{12}$$

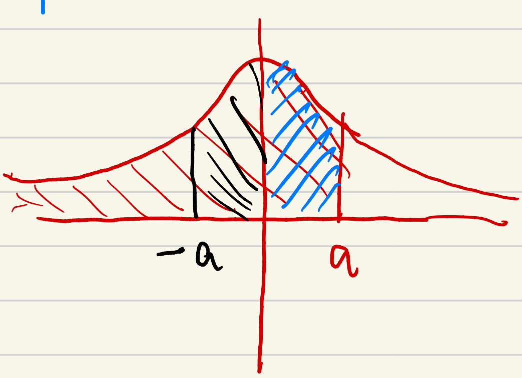
Hence $P\left\{ 29.5 \leq X_1 + \cdots + X_{10} \leq 40.5 \right\}$

$$= P\left\{ \frac{29.5 - 10 \times \frac{7}{2}}{\sqrt{350/12}} \leq \frac{X_1 + \cdots + X_{10} - 10 \times \frac{7}{2}}{\sqrt{10} \cdot \sqrt{\frac{35}{12}}} \leq \frac{40.5 - 10 \times \frac{7}{2}}{\sqrt{350/12}} \right\}$$

$$\approx P\{ -1.018 \leq Z \leq 1.018 \}$$

$$= 2 \cdot \Phi(1.018) - 1$$

$$\approx 0.692.$$



Example 5. Suppose a fair die is rolled for 100 times.

Let X_i be the value obtained in the i-th roll. Compute an approximation of

$$P\left\{ \prod_{i=1}^{100} X_i \leq a^{100} \right\}, \quad 1 < a < 6$$

Sketched solution:

$$P\left\{ \prod_{i=1}^{100} X_i \leq a^{100} \right\}$$

$$= P\left\{ \sum_{i=1}^{100} \log X_i \leq 100 \cdot \log a \right\}$$

(Letting $Y_i = \log X_i$)

$$= P\{ Y_1 + \cdots + Y_{100} \leq 100 \log a \}$$

$$= P\left\{ \frac{Y_1 + \cdots + Y_{100} - 100\, E[Y]}{10 \cdot \sqrt{Var(Y)}} \leq \frac{100 \log a - 100\, E[Y]}{10 \sqrt{Var(Y)}} \right\}$$

$$\approx \Phi\left(\frac{100 \log a - 100\, E[Y]}{10 \sqrt{Var(Y)}} \right)$$

estimate $E[Y]$, $Var[Y]$ and plug them in the expression.

Thm (The central limit Thm).

Let $X_1, \cdots, X_n, \cdots$, be an i.i.d. sequence of r.v.'s, each having finite mean μ and variance σ^2.

Then $\forall\ a \in \mathbb{R}$,

$$P\left\{ \frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma} \leq a \right\} \longrightarrow \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{\frac{-x^2}{2}}\, dx,$$

as $n \to \infty$.

To prove the CLT, we state a result without proof.

Lem 1. Let $Z_1, \cdots, Z_n, \cdots$ be a sequence of r.v's with distribution functions F_{Z_n}. Let Z be a r.v. with distribution function F_Z.

Suppose $M_{Z_n}(t) \to M_Z(t)$ for all $t \in \mathbb{R}$ as $n \to \infty$. (Recall $M_Z(t) := E[e^{tZ}]$)

Then

$$F_{Z_n}(t) \to F_Z(t)$$ for each t at which F_Z is cts, as $n \to \infty$.

Pf of the CLT.

First assume $\mu = 0$, $\sigma^2 = 1$.

Let $$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}, \quad n = 1, 2, \cdots.$$

Let Z be the standard normal r.v.

Recall $M_Z(t) = e^{t^2/2}$, $t \in \mathbb{R}$.

Hence we only need to prove for $t \in \mathbb{R}$,

① $M_{Z_n}(t) \longrightarrow e^{t^2/2}$ as $n \to \infty$.

Notice that

$$M_{Z_n}(t) = E\left[e^{t \cdot \frac{X_1 + \cdots + X_n}{\sqrt{n}}} \right]$$

$$= \prod_{j=1}^{n} E\left[e^{t X_i/\sqrt{n}} \right]$$

$$= \left(M_X\left(\tfrac{t}{\sqrt{n}}\right) \right)^n, \quad \text{where } X = X_1$$

To show ①, it is equivalent to show

② $n \log M_X\left(\frac{t}{\sqrt{n}}\right) \to t^2/2$ as $n \to \infty$.

For convenience, we write

$$L(t) = \log M_X(t).$$

Clearly, $L(0) = 0$.

Notice that

$$L'(t) = \frac{M_X'(t)}{M_X(t)}, \quad L''(t) = \frac{M_X''(t) M_X(t) - (M_X'(t))^2}{M_X(t)^2}$$

In particular

$$L'(0) = \frac{M_X'(0)}{M_X(0)} = \frac{E[X]}{1} = \mu = 0$$

$$L''(0) = \frac{M_X''(0) \cdot M_X(0) - M_X'(0)^2}{M_X(0)^2} = \frac{E[X^2]}{1}$$

$$= Var(X) + E[X]^2$$

$$= 1$$

Hence

$$\lim_{n\to\infty} n\, L\left(\frac{t}{\sqrt{n}}\right) = \lim_{n\to\infty} \frac{L\left(\frac{t}{\sqrt{n}}\right)}{\left(\frac{1}{\sqrt{n}}\right)^2}$$

$$\overset{\text{Letting } x = \frac{1}{\sqrt{n}}}{=} \lim_{x\to 0} \frac{L(tx)}{x^2}$$

$$\overset{\text{L'Hopital's rule}}{=} \lim_{x\to 0} \frac{L'(tx)\cdot t}{2x}$$

$$= \lim_{x\to 0} \frac{L''(tx)\, t^2}{2}$$

$$= \frac{t^2}{2} L''(0) = t^2/2 .$$

In the general case,

$$\frac{X_1+\cdots+X_n-n\mu}{\sqrt{n}\cdot\sigma}=\frac{\frac{X_1-\mu}{\sigma}+\cdots+\frac{X_n-\mu}{\sigma}}{\sqrt{n}}$$

Notice that $\tilde{X}_i=\frac{X_i-\mu}{\sigma}$ has mean 0 and variance 1

Since $\tilde{X}_1,\cdots,\tilde{X}_n,\cdots,$ are i.i.d with mean 0 and variance 1, the distribution of

$\frac{\tilde{X}_1+\cdots+\tilde{X}_n}{\sqrt{n}}$ converges to the standard normal distribution. $\boxed{}$

Thm 3 (The strong law of large numbers).

Let $X_1, \cdots, X_n, \cdots,$ be an i.i.d. sequence of r.v.'s with a finite mean μ. Then with prob. 1,

$$\frac{X_1 + \cdots + X_n}{n} \longrightarrow \mu \qquad \text{as } n \to \infty.$$

In other word,

$$P\left\{ \lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \mu \right\} = 1.$$

Lem 4. Assume X is a non-negative r.v. which may take the value $+\infty$. Suppose that $E[X] < \infty$. Then $P\{X < \infty\} = 1$.

Pf. $P\{X = \infty\} \leq P\{X \geq n\} \leq \frac{E[X]}{n} \to 0.$ ◪

Pf of Thm 3 (SLLN):

We will the thm under an additional assumption

$$E[X_i^4] = K < \infty.$$

WLOG, assume $\mu = 0$.

Write $S_n = X_1 + \cdots + X_n$.

We will estimate

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4].$$

Expand $(X_1 + \cdots + X_n)^4$ in terms of

$$X_i^4, \quad X_i^3 X_j, \quad X_i^2 X_j^2, \quad X_i^2 X_j X_k, \quad X_i X_j X_k X_\ell$$

with distinct i, j, k, ℓ.

Notice that

$$E[X_i^3 X_j] = E[X_i^3]\, E[X_j] = 0$$

$$E[X_i^2 X_j X_k] = E[X_i^2]\, E[X_j]\, E[X_k] = 0$$

$$E[X_i X_j X_k X_\ell] = 0$$

Hence

$$E[S_n^4] = E[(X_1 + \cdots + X_n)^4]$$

$$= n E[X_i^4]$$

$$+ \binom{n}{2}\binom{4}{2} E[X_i^2 X_j^2]$$

$$= n E[X_i^4] + 6\binom{n}{2} E[X_i^2] E[X_i^2]$$

(Using an inequality

$$E[X^2]^2 \leq E[X^4]$$

(reason: $Var(X^2) = E[X^4] - E[X^2]^2 \geq 0$)

$$E[S_n^4] \leq n E[X_i^4] + 6\binom{n}{2} E[X_i^4]$$

$$= \left(n + 6 \cdot \frac{n(n-1)}{2}\right) K$$

$$= (3n^2 - 2n) K.$$

$$\leq 3n^2 K.$$

$$E\left[\frac{S_n^4}{n^4}\right] \leq \frac{3K}{n^2}.$$

Hence

$$\sum_{n=1}^{\infty} E\left[\left(\frac{S_n}{n}\right)^4\right] \leq \sum_{n=1}^{\infty} \frac{3K}{n^2} < \infty.$$

Thus $$E\left[\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4\right] = \sum_{n=1}^{\infty} E\left[\left(\frac{S_n}{n}\right)^4\right] < \infty.$$

Let $X = \sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4$. Then X is a r.v, non-negative (may take the value ∞)

However $E[X] < \infty$

By Lem 4, $P\{X < \infty\} = 1$.

Hence $P\left\{\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4 < \infty\right\} = 1$

However $\sum_{n=1}^{\infty}\left(\frac{S_n}{n}\right)^4 < \infty \Rightarrow \frac{S_n}{n} \to 0$ as $n \to \infty$.

Hence $P\left\{\lim_{n\to\infty}\frac{S_n}{n} = 0\right\} = 1.$

Hence with Prob. 1,

$$\frac{S_n}{n} = \frac{X_1+\cdots+X_n}{n} \to 0 .$$

If $\mu \neq 0$, then letting $\widetilde{X}_n = X_n - \mu$
applying the SLLN to $(\widetilde{X}_n)$ gives

$$\frac{\widetilde{X}_1+\cdots+\widetilde{X}_n}{n} \to 0 \quad \text{almost sure.}$$

$$\Longleftrightarrow \quad \frac{X_1+\cdots+X_n}{n} \to \mu \quad \text{almost sure.}$$

$\square$