# 6.3. Sum of independent R.V.s

1. The case that both $X$ and $Y$ are continuous.

Let $X$, $Y$ have densities $f_X(x)$, $f_Y(y)$ resp.

Since $X$ and $Y$ are assumed to be independent, $X$, $Y$ have a joint density

$$f(x,y) = f_X(x)\, f_Y(y).$$

Now let $a \in \mathbb{R}$, then

$$\begin{aligned}
F_{X+Y}(a) &= P\{X+Y \leq a\} \\
&= \iint_{\substack{(x,y)\in\mathbb{R}^2: \\ x+y \leq a}} f(x,y)\,dx\,dy \\
&= \iint_{(x,y)\in\mathbb{R}^2:\ x+y\leq a} f_X(x)\, f_Y(y)\,dx\,dy \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{a-y} f_X(x)\, f_Y(y)\,dx \right) dy \\
&= \int_{-\infty}^{\infty} f_Y(y) \cdot \left( \int_{-\infty}^{a-y} f_X(x)\,dx \right) dy
\end{aligned}$$

$$= \int_{-\infty}^{\infty} f_Y(y)\, F_X(a-y)\, dy$$

$$=: \quad F_X * f_Y(a)$$

Convolution

( For $g, h : \mathbb{R} \to \mathbb{R}$, we let

$$g * h(a) = \int_{-\infty}^{\infty} g(a-y)\, h(y)\, dy \;)$$

$$f_{X+Y}(a) = \frac{d}{da} F_{X+Y}(a)$$

$$= \frac{d}{da} \int_{-\infty}^{\infty} F_X(a-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left( \frac{d}{da} F_X(a-y) \right) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= f_X * f_Y(a).$$

Thm A. If $X, Y$ are independent, and have densities $f_X(x)$, $f_Y(y)$, then

$$\begin{cases} F_{X+Y}(a) = F_X * f_Y(a). \\ f_{X+Y}(a) = f_X * f_Y(a). \end{cases}$$

Example 1. Let $X, Y$ be independent, both unif. dist. on $[0,1]$. Calculate the density of $X+Y$.

Solution: Let $a \in \mathbb{R}$. Then

$$f_{X+Y}(a) = f_X * f_Y(a)$$

$$= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

$$= \int_0^1 f_X(a-y)\, dy \quad \left(\text{since } f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}\right)$$

$$\overset{\text{letting } z = a-y}{=\!=} \int_{a-1}^{a} f_X(z)\, dz$$

If $0 < a \leq 1$,

$$\int_{a-1}^{a} f_X(z)\,dz = \int_0^a 1\,dz = a.$$

If $1 < a \leq 2$,

$$\int_{a-1}^{a} f_X(z)\,dz = \int_{a-1}^{1} 1\,dz = 2-a.$$

If $a > 2$ or $a < 0$,

$$\int_{a-1}^{a} f_X(z)\,dz = 0.$$

Hence

$$f_{X+Y}(a) = \begin{cases} a & \text{if } 0 < a \leq 1 \\ 2-a & \text{if } 1 < a \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

$\boxtimes$

Example 2. Let $X$, $Y$ be independent normal r.v's with parameters $(0,1)$ and $(0,\sigma^2)$.
Find out the distribution of $X+Y$.

Solution: Recall that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}, \quad y \in \mathbb{R}.$$

Hence for $a \in \mathbb{R}$,

$$f_X * f_Y(a) = \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-y)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{(a-y)^2}{2} - \frac{y^2}{2\sigma^2}} dy.$$

Notice that

$$\frac{(a-y)^2}{2} + \frac{y^2}{2\sigma^2} = \frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2} + \frac{a^2}{2(\sigma^2+1)}$$

( verify it )

Hence

$$f_{X+Y}(a) = \frac{1}{2\pi\sigma} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2}} dy$$

( letting $z = \frac{\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}}{\sigma}$ )

$$= \frac{1}{2\pi\sigma} \cdot \frac{\sigma}{\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

( Using the fact $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$ )

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}} .$$

Hence $X+Y$ is a normal r.v. with parameters $(0, \sigma^2+1)$.

Remark: In general, if $X$, $Y$ are independent, normal r.v.'s with parameters $(\mu_1, \sigma_1^2)$, and $(\mu_2, \sigma_2^2)$, then $X+Y$ has a normal distribution with parameters $(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

Example 1. Let $X$, $Y$ be independent Poisson r.v.'s with parameters $\lambda_1$, $\lambda_2$, resp.
Calculate the distribution of $X+Y$.

Solution: $P\{X=k\} = e^{-\lambda_1} \cdot \lambda_1^k / k!$, $k=0,1,2,\cdots$

$P\{Y=k\} = e^{-\lambda_2} \lambda_2^k / k!$, $k=0,1,2,\cdots$

For $n=0,1,2,\cdots$,

$$P\{X+Y=n\} = \sum_{k=0}^{\infty} P\{X=k\} P\{Y=n-k\}$$

( but $P\{Y=n-k\}=0$ if $k>n$)

$$= \sum_{k=0}^{n} P\{X=k\} P\{Y=n-k\}$$

$$= \sum_{k=0}^{n} e^{-\lambda_1} \frac{\cdot \lambda_1^k}{k!} e^{-\lambda_2} \frac{\lambda_2^{n-k}}{(n-k)!}$$

$$= e^{-\lambda_1-\lambda_2} \cdot \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda_1^k \lambda_2^{n-k}$$

By the binomial Thm

$$= e^{-\lambda_1-\lambda_2} (\lambda_1+\lambda_2)^n / n!$$

( Recall $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$. )

Hence $X+Y$ has ~~the~~ the Poisson distribution with parameter $\lambda_1+\lambda_2$.

## § 6.4 Conditional distribution.

### 1. Discrete case.

**Def.** Let $X, Y$ be two discrete r.v.'s.

Then

$$P\{X=x \mid Y=y\} = \frac{P\{X=x, Y=y\}}{P\{Y=y\}},$$

provided that $P\{Y=y\} > 0$.

2. Suppose that $X$ and $Y$ are jointly cts with density $f(x, y)$.

Def. The $\boxed{\text{conditional density function of } X}$ given $Y=y$, is given by

$$f_{X|Y}(x|y) := \frac{f(x,y)}{f_Y(y)},$$

provided that $f_Y(y) > 0$.

Def. For $A \subset \mathbb{R}$, the conditional prob. of $X$ taking values in $A$ given $Y=y$ is given by

$$P\{X \in A \mid Y=y\} = \int_A f_{X|Y}(x|y)\,dx$$

In particular,

$$F_{X|Y}(a|y) := P\{X \le a \mid Y=y\}$$

$$= \int_{-\infty}^{a} f_{X|Y}(x|y)\,dx.$$

Remark: If $X$ and $Y$ are independent,

then $f_{X|Y}(x|y) = f_X(x)$.

(since in such case $f(x,y) = f_X(x) f_Y(y)$

Remark: One may view

$$P\{X \in A \mid Y = y\}$$

$$= \lim_{\varepsilon \to 0} P\{X \in A \mid y-\varepsilon < Y < y+\varepsilon\}$$

$$= \lim_{\varepsilon \to 0} \frac{P\{X \in A,\ y-\varepsilon < Y < y+\varepsilon\}}{P\{y-\varepsilon < Y < y+\varepsilon\}}$$

Example 2. Suppose the joint density of $X$ and $Y$ is given by

$$f(x,y) = \begin{cases} e^{-x/y} e^{-y}/y & \text{if } x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$$

Find $P\{X>1 \mid Y=y\}$.

Solution: $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^{\infty} e^{-x/y} e^{-y}/y \, dx \quad (\text{if } y>0)$$

$$= -e^{-x/y} e^{-y} \Big|_{x=0}^{\infty}$$

$$= e^{-y} \qquad \text{if } y>0.$$

Hence for $y>0$,

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = e^{-x/y}/y$$

if $x>0$.

Therefore

$$P\{X>1 \mid Y=y\} = \int_1^\infty e^{-x/y}/y \, dx$$

$$= -e^{-x/y} \Big|_{x=1}^{\infty}$$

$$= e^{-1/y} \qquad \text{if } y>0.$$

- Distribution of functions of r.v.'s.

Setup : $X_1$, $X_2$ are joint cts with density $f_{X_1, X_2}(x_1, x_2)$. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$.

Let $Y_1 = g_1(X_1, X_2)$, $Y_2 = g_2(X_1, X_2)$.

Find out the joint distribution of $Y_1$, $Y_2$

Thm. Assumptions: ① $x_1, x_2$ can be solved in terms of $y_1, y_2$.

② $g_1, g_2$ have cts partial derivatives

and

$$J(x_1, x_2) := \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{vmatrix}$$

$$= \frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_1}{\partial x_1} \neq 0$$

Then $Y_1, Y_2$ have a joint density

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(x_1, x_2) \cdot |J(x_1, x_2)|^{-1}.$$

Exer 1. $X, Y$ have joint density

$$f(x,y) = \begin{cases} \frac{1}{x^2y^2}, & \text{if } x>1, \ y>1 \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = XY$, $V = X/Y$. Find out the joint density of $U$ and $V$.

Solution: Let $g_1(x,y) = xy$ and $g_2(x,y) = x/y$.

Then

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix}$$

$$= -\frac{2x}{y}$$

By the thm,

$$f_{U,V}(u,v) = f(x,y) \cdot |J(x,y)|^{-1}$$

$$= \begin{cases} \frac{1}{x^2y^2} \cdot \frac{y}{2x} = \frac{1}{2x^3y} & \text{if } x>1,\ y>1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $x = \sqrt{uv}$

$y = \sqrt{u/v}$

$u, v > 0.$

$\left( \begin{array}{l} \sqrt{uv} > 1 \Rightarrow v > \frac{1}{u} \\ \sqrt{u/v} > 1 \Rightarrow v < u \end{array} \right.$

Also notice that $x, y > 1 \iff u > 1,\ \frac{1}{u} < v < u.$

Hence $f_{U,V}(u,v) = \begin{cases} \frac{1}{2u^2v}, & \text{if } u>1,\ \frac{1}{u}<v<u \\ 0, & \text{otherwise.} \end{cases}$ $\square$