Chapter 6. Joint distributed R.V.s

Def. Let X, Y be two r.v.'s.

The joint cumulative distribution function of X, Y (joint CDF) is defined by

$$F(a,b) = P\{X \leq a, Y \leq b\}, \quad a, b \in \mathbb{R}.$$

From the above def, we see that F_X and F_Y can be obtained from $F(\cdot,\cdot)$.

$$F_X(a) = P\{X \leq a\}$$
$$= P\{X \leq a, Y < \infty\}$$
$$= P\left(\lim_{b \to +\infty} \{X \leq a, Y \leq b\}\right)$$

Using the continuity of P

$$= \lim_{b \to +\infty} P\{X \leq a, Y \leq b\}$$
$$= \lim_{b \to +\infty} F(a,b)$$
$$=: F(a, +\infty)$$

Similary, $F_Y(b) = \lim_{a \to +\infty} F(a,b) =: F(+\infty, b)$.

- Usually, F_X and F_Y are called the marginal distributions of X and Y.

Theoretically, all the joint statements about X and Y can be determined by the function $F(a,b)$.

Example 4. Suppose $F = F(a,b)$ is the joint CDF of X and Y. Find $P\{X > a, Y > b\}$.

Solution:

$$P\{X>a, Y>b\} = 1 - P\left(\{X>a, Y>b\}^c\right)$$

$$= 1 - P\left(\{X \le a\} \cup \{Y \le b\}\right)$$

(using $P(E \cup F) = P(E) + P(F) - P(EF)$)

$$= 1 - P\{X \le a\} - P\{Y \le b\} + P\{X \le a, Y \le b\}$$

$$= 1 - F(a, \infty) - F(\infty, b) + F(a,b). \quad \square$$

(1) Discrete case.

- Now we consider the case that both X and Y are discrete. In such case, we can define the joint prob. mass function of X and Y by (joint pmf)

$$p(x,y) = P\{X=x, Y=y\}.$$

Then

$$\begin{aligned} p_X(x) &= P\{X=x\} \\ &= \sum_y P\{X=x, Y=y\} \\ &= \sum_y p(x,y). \end{aligned}$$

Similarly

$$p_Y(y) = \sum_x p(x,y).$$

In particular

$$F(a,b) = \sum_{\substack{(x,y) \\ x\le a,\, y\le b.}} p(x,y).$$

(2) Continuous case.

- Def: We say two r.v's X and Y are jointly continuous if there exists $f: \mathbb{R}^2 \to [0,\infty)$ such that

$$P\{(X,Y)\in C\} = \iint_C f(x,y)\,dx\,dy,$$

for any "measurable" set $C \subset \mathbb{R}^2$.

("measurable" sets include, for instance, the countable union/intersections of rectangles $[a,b]\times[c,d]$)

- In particular,

$$P\{X\le a, Y\le b\} = P\{(X,Y)\in(-\infty,a)\times(-\infty,b)\}$$
$$= \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

- The function f in the def is called the joint prob. density function of X and Y.

Prop 1. Suppose X and Y have a joint density f.
Let F be the joint CDF of X and Y, and let f_X and f_Y be the marginal densities of X and Y.

Then

(1) $$\frac{\partial^2 F(a,b)}{\partial a \partial b} = f(a,b) \text{ for } a, b \in \mathbb{R}.$$

(2) $$f_X(a) = \int_{-\infty}^{\infty} f(a,y)\, dy, \qquad a \in \mathbb{R}$$

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\, dx, \qquad b \in \mathbb{R}.$$

Pf. (1) Recall that

$$F(a,b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{a} g(x)\, dx \quad \text{where } g(x) = \int_{-\infty}^{b} f(x,y)\, dy$$

So $$\frac{\partial F(a,b)}{\partial a} = g(a) = \int_{-\infty}^{b} f(a,y)\, dy$$

and $$\frac{\partial^2 F(a,b)}{\partial a \partial b} = \frac{\partial}{\partial b} \int_{-\infty}^{b} f(a,y)\, dy = f(a,b).$$

(2)

$$F_X(a) = P\{X \leq a\}$$
$$= \int_{-\infty}^{a} \left(\int_{-\infty}^{\infty} f(x,y)\,dy \right) dx$$

Let $h(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Then

$$F_X(a) = \int_{-\infty}^{a} h(x)\,dx$$

Taking derivatives gives

$$f_X(a) = \frac{d F_X(a)}{da} = h(a) = \int_{-\infty}^{\infty} f(a,y)\,dy.$$

Similarly

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx.$$

$\boxed{/\!/}$

Example 2. Suppose X and Y have a joint density function

$$f(x,y) = \begin{cases} 12\, xy\,(1-x) & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Find f_X and $E[X]$.

Solution: By Prop 1,

$$f_X(a) = \int_{-\infty}^{\infty} f(a,y)\, dy .$$

For $a \in (0,1)$,

$$f_X(a) = \int_0^1 12a(1-a)\, y\, dy$$

$$= 6a(1-a).$$

For $a \notin (0,1)$, $f_X(a) = 0$.

Hence $f_{X(a)} = \begin{cases} 6a(1-a) & \text{if } a \in (0,1) \\ 0 & \text{otherwise.} \end{cases}$

Now

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^1 6x^2(1-x)\, dx$$

$$= \int_0^1 6x^2 - 6x^3\, dx$$

$$= 2x^3 - \frac{3}{2}x^4 \Big|_0^1$$

$$= \frac{1}{2}.$$

Example 3.

Suppose X and Y have a joint density function

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the prob. density function of $\frac{X}{Y}$.

Solution:

Since $f(x,y) = 0$ if $(x,y) \notin (0,\infty) \times (0,\infty)$, we may assume X, Y always take positive values. So is X/Y.

For $a > 0$,

$$P\left\{ \frac{X}{Y} \le a \right\} = P\{ X \le aY \}$$

$$= \iint_{\{(x,y):\ x \le ay\}} f(x,y)\, dx\, dy$$

$$= \iint_{\{(x,y)\in(0,\infty)\times(0,\infty)\,:\,x\le ay\}} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty \int_0^{ay} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty e^{-y}\cdot\left(-e^{-x}\right)\Big|_0^{ay}\,dy$$

$$= \int_0^\infty e^{-y}\cdot\left(1-e^{-ay}\right)\,dy$$

$$= \int_0^\infty e^{-y} - e^{-(a+1)y}\,dy$$

$$= -e^{-y} + \frac{1}{1+a} e^{-(a+1)y}\Big|_0^\infty$$

$$= 1 - \frac{1}{1+a}$$

Taking derivative gives

$$f_{\frac{X}{Y}}(a) = \begin{cases} \frac{1}{(1+a)^2} & a > 0 \\ 0 & \text{otherwise}. \end{cases}$$

$\boxtimes$

§ 6.2 Independent random variables

Recall that two events E and F are said to be independent if $P(E \cap F) = P(E)P(F)$.

Def: Let X and Y be two r.v.'s.

We say that X and Y are independent if

$$P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\},$$

for all $A, B \subset \mathbb{R}$. That is, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for all $A, B \subset \mathbb{R}$.

Remark: X and Y are independent

$$\Longleftrightarrow$$

$$F(a,b) = F_X(a) F_Y(b), \quad \forall\, a, b \in \mathbb{R}.$$

The direction "$\Rightarrow$" is clear. The other direction can be proved by using the three axioms of probability.

- Equivalent def of independence for r.v.'s

Prop 5. Suppose X and Y are discrete. Then

X and Y are independent

$$\iff \quad p(x,y) = p_X(x)\, p_Y(y) \qquad (*)$$

Pf. Clearly X and Y are independent

$$\iff P\{X\in A, Y\in B\} = P\{X\in A\}\cdot P\{Y\in B\}.$$

Letting $A=\{x\}$, $B=\{y\}$ gives

$$p(x,y) = p_X(x)\, p_Y(y).$$

Now suppose $(*)$ holds for all x, y,

Then for given $A, B \subset \mathbb{R}$,

$$\begin{aligned} P\{X\in A, Y\in B\} &= \sum_{x\in A}\sum_{y\in B} p(x,y) \\ &= \sum_{x\in A}\sum_{y\in B} p_X(x)\, p_Y(y) \\ &= \Big(\sum_{x\in A} p_X(x)\Big)\Big(\sum_{y\in B} p_Y(y)\Big) \\ &= P\{X\in A\}\, P\{Y\in B\}. \quad \square \end{aligned}$$

Prop 6. If X and Y are jointly continuous, then X and Y are independent

$$\Leftrightarrow \quad f(x,y) = f_X(x) f_Y(y).$$

Pf. X and Y are independent

$$\Leftrightarrow \quad F(a,b) = F_X(a) F_Y(b), \quad \forall a,b \in \mathbb{R}$$

$$\Rightarrow \quad \frac{\partial^2 F(a,b)}{\partial a \partial b} = \frac{d F_X(a)}{d a} \cdot \frac{d F_Y(b)}{d b}$$

i.e. $\quad f(a,b) = f_X(a) f_Y(b). \quad (**)$

Now if $(**)$ holds, then

$$\begin{aligned} F(a,b) &= \int_{-\infty}^{b} \int_{-\infty}^{a} f(x,y)\,dx\,dy \\ &= \int_{-\infty}^{b} \int_{-\infty}^{a} f_X(x) f_Y(y)\,dx\,dy \\ &= \left(\int_{-\infty}^{b} f_Y(y)\,dy\right) \left(\int_{-\infty}^{a} f_X(x)\,dx\right) \\ &= F_Y(b) \cdot F_X(a). \end{aligned}$$

Hence X, Y are independent. $\square$

Example 7: Suppose X and Y have a joint density

$$f(x,y) = 24xy, \quad \text{if } 0<x<1,\ 0<y<1,\ 0<x+y<1.$$

Determine whether X and Y are independent.

Solution: We first calculate the marginal densities $f_X(x)$, $f_Y(y)$.

Notice that for $0<a<1$,

$$f(a,y) = \begin{cases} 24ay, & \text{if } 0<y<1-a, \\ 0 & \text{otherwise.} \end{cases}$$

So $$f_X(a) = \int_{-\infty}^{-\infty} f(a,y)\,dy$$

$$= \int_0^{1-a} 24ay\,dy$$

$$= 24a\,\frac{y^2}{2}\Big|_0^{1-a} = 12a\cdot(1-a)^2$$

Similarly,

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx$$

$$= \int_0^{1-b} 24\,x\,b\,dx$$

$$= 12\,b\,(1-b)^2 \qquad \text{if } 0<b<1.$$

Clearly $f(a,b) \neq f_X(a)\,f_Y(b)$. Hence X, Y are not independent. $\blacksquare$