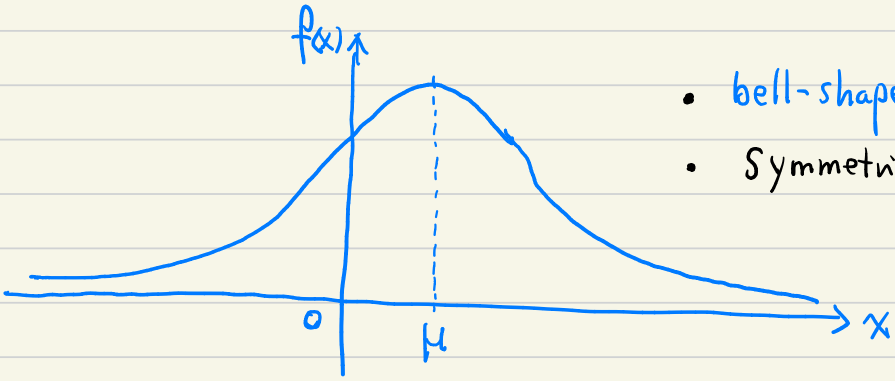
§ 5.4 Normal r.v.

Def. Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Say X is a normal r.v. with parameters μ and σ^2 if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$



- bell-shaped
- Symmetric w.r.t. $x = \mu$

Below we give some facts about normal r.v.'s

Fact 1

The above f is indeed a pdf.

To see this,

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \underset{\text{Letting } y=\frac{x-\mu}{\sigma}}{=\!=} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy$$

We need to show $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy = \sqrt{2\pi}$. Notice

$$(*)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy$$

$$\underset{\text{Letting } \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}}{=\!=} \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\int_0^{\infty} e^{-\frac{r^2}{2}}\, r\,dr\right) d\theta$$

$$= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}}\, r\,dr$$

$$= 2\pi \left(-e^{-\frac{r^2}{2}}\right)\Big|_0^{\infty}$$

$$= 2\pi.$$

Since $(*) = \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy\right)^2$, we obtain the desired identity.

Fact 2

Let X be a normal r.v with parameters μ and σ^2, then

$$Z := \frac{X-\mu}{\sigma}$$

is a normal r.v with parameters 0 and 1.

We call Z a standard normal r.v.

Pf. Calculate the cumulative distribution of Z,

$$\begin{aligned} F_Z(a) &= P\{Z \leq a\} \\ &= P\left\{\frac{X-\mu}{\sigma} \leq a\right\} \\ &= P\{X \leq a\sigma + \mu\} \\ &= \int_{-\infty}^{a\sigma+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\ &\overset{\text{Letting } y=\frac{x-\mu}{\sigma}}{=} \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \end{aligned}$$

Taking derivative of $F_Z(a)$ gives

$$f_Z(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}.$$

Hence Z is the standard normal r.v.

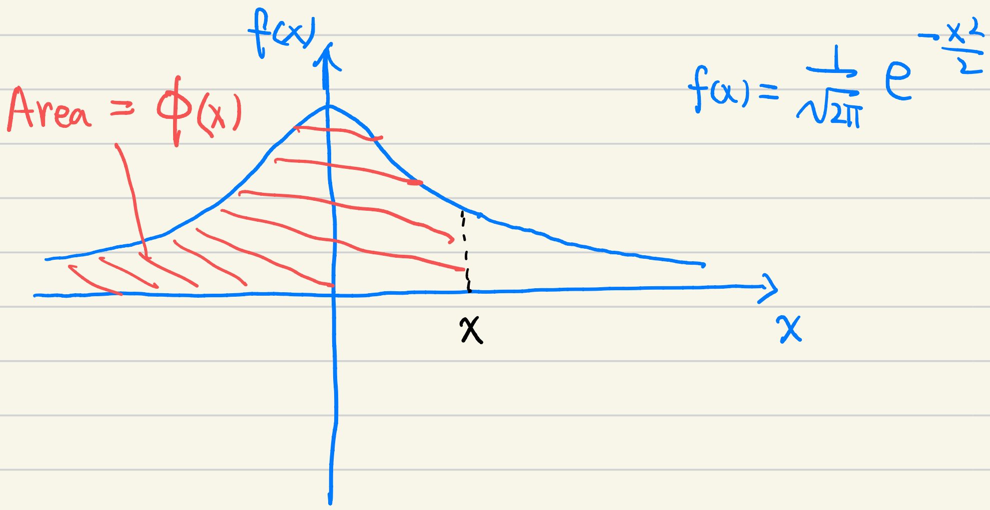
- Let

$$\phi(x) = P\{Z \leq x\}$$

be the CDF of Z.

Then

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$



Fact 3 $\quad E[Z]=0$ and $V(Z)=1$.

Pf.

$$E[Z] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty}$$

$$= 0$$

$$E[Z^2] = \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot x \cdot \left(-e^{-\frac{x^2}{2}}\right)'\, dx$$

int by parts

$$= \frac{1}{\sqrt{2\pi}} \cdot x \cdot \left(-e^{-\frac{x^2}{2}}\right) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot (x)'\, dx$$

$$= 1$$

Hence

$$V(Z) = E[Z^2] - (E[Z])^2$$
$$= 1 - 0^2 = 1. \quad \square$$

Fact 4. Let $Y = aX + b$, where $a, b \in \mathbb{R}$ and X is a cts r.v.

Then ① $E[Y] = a\,E[X] + b$

② $V(Y) = a^2 V(X)$.

Pf. Let f be the pdf of X.

Then

$$E[Y] = \int_{-\infty}^{\infty} (ax+b) f(x)\,dx$$
$$= a\int_{-\infty}^{\infty} x f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$
$$= a\,E[X] + b$$

$$E[Y^2] = \int_{-\infty}^{\infty} (ax+b)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} (a^2x^2 + 2abx + b^2) f(x)\, dx$$

$$= a^2\int_{-\infty}^{\infty} x^2 f(x)\, dx + 2ab \int_{-\infty}^{\infty} x f(x)\, dx$$

$$+ b^2 \int_{-\infty}^{\infty} f(x)\, dx$$

$$= a^2 E[X^2] + 2ab\, E[X] + b^2$$

Hence

$$V(Y) = E[Y^2] - (E[Y])^2$$

$$= a^2 E[X^2] + 2ab E[X] + b^2$$

$$- (aE[X] + b)^2$$

$$= a^2 (E[X^2] - (E[X])^2)$$

$$= a^2 V(X).$$

$\square$

Fact 5. Let X be a normal r.v. with parameters μ and σ^2.

Then $E[X] = \mu$

$V(X) = \sigma^2$.

Pf. Let $Z = \frac{X-\mu}{\sigma}$. Then Z is a standard normal r.v. Hence $E[Z]=0$, $V(Z)=1$.

Now $X = \sigma Z + \mu$. By Lem 1,

$$E[X] = \sigma E[Z] + \mu = \sigma \cdot 0 + \mu = \mu.$$

$$V(X) = \sigma^2 V(Z) = \sigma^2 \cdot 1 = \sigma^2. \quad \square$$

The most important property of normal r.v is that it can be used to approximate a binomial r.v with parameters (n, p) when n is large.

Thm 1 (DeMoivre-Laplace Thm)

Let $0<p<1$. Let X_n be a binomial r.v with parameters (n,p). Then for given $a, b \in \mathbb{R}$,

$$P\left\{ a < \frac{X_n - np}{\sqrt{np(1-p)}} < b \right\}$$

$$\xrightarrow{\text{as } n\to\infty} P\{ a < Z < b \}$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \phi(b) - \phi(a).$$

- In other word, for large n,

$\frac{X_n - nP}{\sqrt{nP(1-P)}}$ has approximately the same distribution as Z.

Example 2. Let X be a binomial r.v with parameters $n = 40$ and $p = \frac{1}{2}$.

Find $P\{X = 20\}$.

Solution: We first find the precise value of this prob.

$$P\{X = 20\} = \binom{40}{20} \cdot \left(\frac{1}{2}\right)^{40}$$

$$\approx .1254.$$

Next we estimate this prob. by using normal distribution.

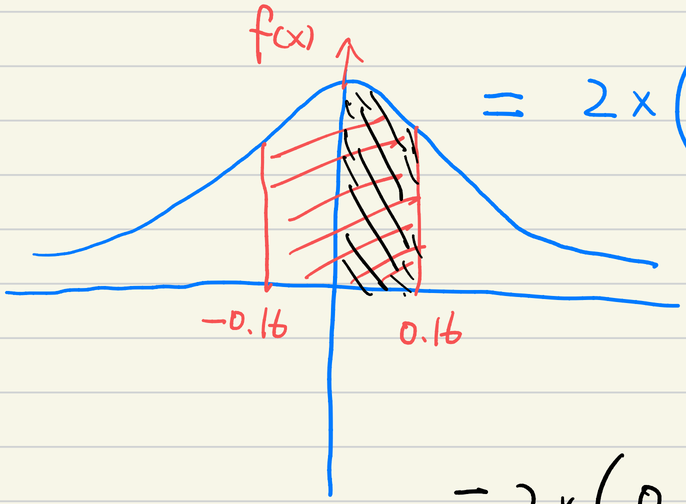
$$P\{X = 20\} = P\{19.5 \leq X \leq 20.5\}$$

(continuity correction)

$$= P\left\{\frac{-0.5}{\sqrt{10}} \leq \frac{X - 20}{\sqrt{10}} \leq \frac{0.5}{\sqrt{10}}\right\}$$

$$\approx P\left\{ \frac{-0.5}{\sqrt{10}} \le Z \le \frac{0.5}{\sqrt{10}} \right\}$$

$$\approx P\{-0.16 \le Z \le 0.16\}$$



$$= 2 \times \left(\Phi(0.16) - \frac{1}{2} \right)$$

$$= 2 \times \left(0.5636 - \frac{1}{2} \right)$$

$$= .1274.$$

Table 5.1 Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X.

$\phi(0.16)$

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

The values of $\Phi(x)$ for nonnegative x are given in Table 5.1. For negative values of x, $\Phi(x)$ can be obtained from the relationship

$$\Phi(-x) = 1 - \Phi(x) \qquad -\infty < x < \infty \tag{4.1}$$

The proof of Equation (4.1), which follows from the symmetry of the standard normal density, is left as an exercise. This equation states that if Z is a standard normal random variable, then

$$P\{Z \leq -x\} = P\{Z > x\} \qquad -\infty < x < \infty$$

§ 5.5 Exponential r.v.

Def. Let $\lambda > 0$. Say X is an exponential r.v. with parameter λ if X has the following density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example3. Find $E[X]$ and $V(X)$

Solution:

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x \;\lambda e^{-\lambda x}\, dx$$

$$= \int_{0}^{\infty} x \cdot \left(-e^{-\lambda x}\right)' dx$$

$$\overset{\text{int by parts}}{=} x \cdot \left(-e^{-\lambda x}\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda x} \cdot (x)'\, dx$$

$$= 0 + \int_{0}^{\infty} e^{-\lambda x} dx$$

$$= 0 + \frac{1}{\lambda} \int_{0}^{\infty} \lambda e^{-\lambda x} dx$$

$$= \frac{1}{\lambda}$$

Notice that

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{\infty} x^2 \cdot \left(-e^{-\lambda x}\right)' dx$$

$$\overset{\text{int by parts}}{=} x^2 \cdot \left(-e^{-\lambda x}\right)\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} \cdot (x^2)'\, dx$$

$$= 0 + \int_0^{\infty} 2x\, e^{-\lambda x} dx$$

$$= 0 + \frac{2}{\lambda} \int_0^{\infty} x \cdot \lambda e^{-\lambda x} dx$$

$$= \frac{2}{\lambda} E[X].$$

$$= \frac{2}{\lambda^2}.$$

Hence $\mathrm{Var}(X) = E[X^2] - E[X]^2$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}.$$

Exer 4. Let X be an exponential r.v with parameter λ. Show that

$$P\{X > s+t \mid X > t\} = P\{X > s\}$$

for any $s, t > 0$.

Pf. For any $a > 0$,

$$P\{X > a\} = \int_a^{+\infty} f(x)\,dx$$

$$= \int_a^{+\infty} \lambda e^{-\lambda x}\,dx$$

$$= -e^{-\lambda x}\Big|_a^{+\infty}$$

$$= e^{-\lambda a}.$$

Write $E = \{X > s+t\}$ and $F = \{X > t\}$.

Then $E \cap F = \{X > s+t\}$.

Hence

$$P\{X > s+t \mid X > t\}$$

$$= P(E \mid F)$$

$$= \frac{P(E \cap F)}{P(F)}$$

$$= \frac{P(E)}{P(F)}$$

$$= P\{X > s+t\} \Big/ P\{X > t\}$$

$$= e^{-\lambda(s+t)} \Big/ e^{-\lambda t}$$

$$= e^{-\lambda s}$$

$$= P\{X > s\}. \qquad \blacksquare$$

§ 5.7 The distribution of a function of a cts r.v

Q: Let X be a cts r.v. with density f_X

Let $g: \mathbb{R} \to \mathbb{R}$ be a function.

Let $Y = g(X)$.

How to find the distribution of Y ?

Example 1. Let X be a cts r.v. with density f_X.

Let $Y = X^2$.

Find the pdf of Y.

Solution: We first calculate the CDF of Y:

$$F_Y(a) = P\{Y \le a\}$$
$$= P\{X^2 \le a\}$$
$$= \begin{cases} 0 & \text{if } a < 0 \\ P\{-\sqrt{a} \le X \le \sqrt{a}\} & \text{if } a \ge 0 \\ = F_X(\sqrt{a}) - F_X(-\sqrt{a}) & \end{cases}$$

Taking derivative of F_Y with respect to a gives

$$f_Y(a) = \begin{cases} 0 & \text{if} \quad a \le 0 \\ \dfrac{f_X(\sqrt{a}) + f_X(-\sqrt{a})}{2\sqrt{a}} & \text{if} \quad a > 0 \end{cases}.$$

$\boxed{//}$

Prop. 2 : Let X be a cts r.v. with density f_X. Let $g: \mathbb{R} \to \mathbb{R}$ be a differentiable, strictly monotone function. Let $Y = g(X)$, then the pdf of Y is given by

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \dfrac{d\, g^{-1}(y)}{dy} \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

pf. WLOG assume that g is strictly monotone increasing.

First suppose that $y = g(x)$ for some x. Then

$$F_Y(y) = P\{ Y \le y \}$$
$$= P\{ g(X) \le y \}$$
$$= P\{ X \le g^{-1}(y) \}$$
$$= F_X(g^{-1}(y)).$$

Taking derivative with respect to y gives

$$f_Y(y) = F_Y'(y) = f_X(g^{-1}(y)) \cdot \frac{d\, g^{-1}(y)}{dy}.$$

If $y \notin \text{range}(g)$, then either

$$F_Y(y) = 0 \text{ or } 1.$$

Taking derivative gives

$$f_Y(y) = 0.$$

$\blacksquare$

Example 3. Let Z be a standard normal r.v. Find the density of Z^3.

Solution: Let $g(x) = x^3$. Then g is monotone increasing on $\mathbb{R}$ with $\text{range}(g) = \mathbb{R}$.

Hence

$$f_{Z^3}(z) = f_Z(g^{-1}(z)) \cdot \frac{d\, g^{-1}(z)}{dz}$$

$$= f_Z(z^{1/3}) \cdot \frac{1}{3} \cdot z^{-2/3}$$

$$= \frac{1}{3\sqrt{2\pi}} \cdot z^{-2/3} \cdot e^{-\frac{z^{2/3}}{2}}.$$

$\blacksquare$