THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3280 Introductory Probability 2024-2025 Term 2
Homework Assignment 3
Due Date: 7 March, 2024 (Friday)

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ______________________ Date ______________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests there as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to 2024R2 Introductory Probability (MATH3280B)
  2. Choose Tools in the left-hand column
  3. Scroll down to the bottom of the page
  4. The green Gradescope icon will be there

- Late assignments will receive a grade of 0.

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

  For the declaration sheet:

  Either

Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your solutions on A4 white paper or use an iPad or similar device to present your answers and submit them in digital form via Gradescope. Please do not use colored paper and ensure that your written solutions are of a suitable size for easy reading. Be aware that you can only use a ball-point pen to write your answers for any exams.
- To receive full credit, all work must be shown. In most cases, a correct answer without supporting work will NOT receive full credit. What you write down and how you write it are crucial for earning good marks on this homework. Neatness and organization are also essential.

1. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings.

   (a) What are the possible values of X, and what are the probabilities associated with each value?

   (b) If we play the game 100 times, and to play every time we have to pay \$2 as table money, what is the amount we should expect to get?

   (c) Is the game fair? Explain.

2. Let an urn contain 7 balls, numbered from 1 to 7. Two balls are selected randomly from the urn. Let X be the difference of the numbers (that is, the largest number minus the smallest number) of the selected balls. Calculate the probability distribution of X, and its mean and standard deviation under the following conditions:

   (a) The balls are selected from the urn with replacement.

   (b) The balls are selected from the urn without replacement.

3. Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \dfrac{|x|}{C} & \text{if } x = -2, -1, 0, 1, 2, \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find C.

   (b) Find $E[X]$.

   (c) Find the PMF of the random variable $Z = (X - E[X])^2$.

   (d) Using Part (c), compute the variance of X.

   (e) Compute the variance of X using the identity $\mathrm{Var}(X) = \sum_x (x - E[X])^2 p_X(x)$.

4. Suppose that the distribution function of X is given by

$$F(b) = \begin{cases} 0 & b < 0; \\ b/4 & 0 \le b < 1; \\ 1/2 + (b-1)/4 & 1 \le b < 2; \\ 11/12 & 2 \le b < 3; \\ 1 & 3 \le b \end{cases}$$

   (a) Find $P(X = i),\ i = 1, 2, 3$.

   (b) Find $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$.

5. When coin 1 is flipped, it lands on heads with probability .4; when coin 2 is flipped, it lands on heads with probability .7. One of these coins is randomly chosen and flipped 10 times.

(a) What is the probability that the coin lands on heads for exactly 7 of the 10 flips?

(b) Given that the first of these ten flips lands on heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

6. An interviewer is given a list of people she can interview. If the interviewer needs to interview 5 people, and if each person (independently) agrees to be interviewed with probability 2/3,

   (a) For part (a), what is the probability that her list of people will enable her to obtain her necessary number of interviews if the list consists of (a) 5 people and (b) 8 people?

   (b) For part (b), what is the probability that the interviewer will speak to exactly (c) 6 people and (d) 7 people on the list?

   Note that there are four subproblems in this question.

7. An absentminded worker does not remember which of his 12 keys will open his office door. If he tries them at random and with replacement:

   (a) On average, how many keys should he try before his door opens?

   (b) What is the probability that he opens his office door after only three tries?

8. Let X be such that
$$P(X = 1) = p = 1 - P(X = -1)$$
   Find $c \neq 1$ such that $E[c^X] = 1$.

9. Let X be a binomial random variable with parameters n and p. Show that
$$E\left[\frac{1}{X+1}\right] = \frac{1-(1-p)^{n+1}}{(n+1)p}.$$

10. Let X be a Poisson random variable with parameter λ. What value of λ maximizes $P(X = k)$, $k \geq 0$?

11. Show that X is a Poisson random variable with parameter λ, then
$$E\left[X^n\right] = \lambda E\left[(X+1)^{n-1}\right]$$
    Now use this result to compute $E\left[X^3\right]$.

12. Let X be a negative binomial random variable with parameters r and p and let Y be a binomial random variable with parameters n and p. Show that
$$P(X > n) = P(Y < r).$$

13. A random variable X is said to follow the hypergeometric distribution with parameters N, M and n if it assumes only non-negative integer values and its probability mass function is given by

$$P(X = x) = \begin{cases} \dfrac{C_x^M \cdot C_{n-x}^{N-M}}{C_n^N} & \text{for } x = 0, 1, 2, \cdots, \min\{n, M\} \\ 0 & \text{otherwise} \end{cases}$$

where n, M, and N are positive integers such that $n \leq N$, and $M \leq N$.

Show whether or not

(a) $E[X] = \dfrac{nM}{N}$

(b) $E[X^2] = \dfrac{M(M-1)n(n-1)}{N(N-1)} + \dfrac{nM}{N}$