MATH2010 Advanced Calculus I

Solution to Homework 7

§13.4

Q6

Solution.

(a) By using the Chain Rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = (-y\cos xy)(1) + (-x\cos xy)(\frac{1}{t}) + (1)(e^{t-1})$$

$$= -(\ln t)\cos(t\ln t) + (-t)\cos(t\ln t)\frac{1}{t} + e^{t-1} = -(\ln t)\cos(t\ln t) - \cos(t\ln t) + e^{t-1}$$

By expressing w in terms of t and differentiating directly with respect to t,

$$\frac{dw}{dt} = \frac{d}{dt}(e^{t-1} - \sin(t\ln t)) = e^{t-1} - \cos(t\ln t)(t\frac{1}{t} + \ln t) = e^{t-1} - (1+\ln t)\cos(t\ln t)$$

(b) $$\left.\frac{dw}{dt}\right|_{t=1} = e^{1-1} - (1+\ln 1)\cos(1\ln 1) = e^0 - (1+0)\cos 0 = 1 - (1)(1) = 0$$

□

Q10

Solution.

(a) By using the Chain Rule,

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= \left(\frac{2x}{x^2+y^2+z^2}\right)(ue^v\cos u + e^v\sin u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(-ue^v\sin u + e^v\cos u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(e^v)$$

$$= \left(\frac{2ue^v\sin u}{(ue^v\sin u)^2 + (ue^v\cos u)^2 + (ue^v)^2}\right)(ue^v\cos u + e^v\sin u)$$

$$+ \left(\frac{2ue^v\cos u}{(ue^v\sin u)^2 + (ue^v\cos u)^2 + (ue^v)^2}\right)(-ue^v\sin u + e^v\cos u) + \left(\frac{2ue^v}{(ue^v\sin u)^2 + (ue^v\cos u)^2 + (ue^v)^2}\right)(e^v)$$

$$= \frac{2}{u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$= \left(\frac{2x}{x^2+y^2+z^2}\right)(ue^v\sin u) + \left(\frac{2y}{x^2+y^2+z^2}\right)(ue^v\cos u) + \left(\frac{2z}{x^2+y^2+z^2}\right)(ue^v)$$

$$= \left(\frac{2ue^v\sin u}{(ue^v\sin u)^2 + (ue^v\cos u)^2 + (ue^v)^2}\right)(ue^v\sin u)$$

$$+\left(\frac{2ue^v\cos u}{(ue^v\sin u)^2+(ue^v\cos u)^2+(ue^v)^2}\right)(ue^v\cos u)+\left(\frac{2ue^v}{(ue^v\sin u)^2+(ue^v\cos u)^2+(ue^v)^2}\right)(ue^v)$$
$$= 2$$

By expressing w directly in terms of u and v before differentiating,
$w = \ln\left((ue^v\sin u)^2 + (ue^v\cos u)^2 + (ue^v)^2\right) = \ln\left(2u^2e^{2v}\right) = \ln 2 + 2\ln u + 2v.$

Hence, $\dfrac{\partial w}{\partial u} = \dfrac{2}{u}, \dfrac{\partial w}{\partial v} = 2.$

(b) $\left.\dfrac{\partial w}{\partial u}\right|_{(u,v)=(-2,0)} = \dfrac{2}{-2} = -1, \left.\dfrac{\partial w}{\partial v}\right|_{(u,v)=(-2,0)} = 2$

□

Q12

Solution.

(a) By using the Chain Rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$$
$$= \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) + (re^{qr}\sin^{-1}p)(0) + (qe^{qr}\sin^{-1}p)(0) = \frac{e^{qr}}{\sqrt{1-p^2}}(\cos x) = \frac{e^{(z^2\ln y)\frac{1}{z}}\cos x}{\sqrt{1-(\sin x)^2}} = y^z \text{ if } -\tfrac{\pi}{2} < x < \tfrac{\pi}{2}$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$
$$= \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1}p)(z^2\frac{1}{y}) + (qe^{qr}\sin^{-1}p)(0) = (re^{qr}\sin^{-1}p)(z^2\frac{1}{y}) = \left(\frac{1}{z}y^z x\right)(z^2\frac{1}{y}) = xy^{z-1}z$$
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$
$$= \frac{e^{qr}}{\sqrt{1-p^2}}(0) + (re^{qr}\sin^{-1}p)(2z\ln y) + (qe^{qr}\sin^{-1}p)(-\frac{1}{z^2}) = \left(\frac{1}{z}y^z x\right)(2z\ln y) + (z^2(\ln y)y^z x)(-\frac{1}{z^2}) = xy^z\ln y$$

By expressing u directly in terms of x, y and z before differentiating,
$u = y^z x$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Hence, $\dfrac{\partial u}{\partial x} = y^z, \dfrac{\partial u}{\partial y} = zy^{z-1}x = xy^{z-1}z, \dfrac{\partial u}{\partial z} = (\ln y)y^z x = xy^z\ln y.$

(b) $\left.\dfrac{\partial u}{\partial x}\right|_{(x,y,z)=(\frac{\pi}{4},\frac{1}{2},-\frac{1}{2})} = \left(\dfrac{1}{2}\right)^{-\frac{1}{2}} = \sqrt{2},$

$\left.\dfrac{\partial u}{\partial y}\right|_{(x,y,z)=(\frac{\pi}{4},\frac{1}{2},-\frac{1}{2})} = (\dfrac{\pi}{4})(\dfrac{1}{2})^{-\frac{1}{2}-1}(-\dfrac{1}{2}) = -\dfrac{\pi\sqrt{2}}{4},$

$\left.\dfrac{\partial u}{\partial z}\right|_{(x,y,z)=(\frac{\pi}{4},\frac{1}{2},-\frac{1}{2})} = (\dfrac{\pi}{4})(\sqrt{2})\ln\left(\dfrac{1}{2}\right) = -\dfrac{\pi\sqrt{2}\ln 2}{4}$

□

Q26

Solution.
Let $F(x,y) = xy + y^2 - 3x - 3$.
$F_x(x,y) = y - 3, F_y(x,y) = x + 2y$.
Then, $\frac{dy}{dx} = -\frac{Fx}{Fy} = -\frac{y-3}{x+2y}$.
$\left.\frac{dy}{dx}\right|_{(x,y)=(-1,1)} = -\frac{1-3}{-1+2} = 2$ □

Q30

Solution.
$xe^{x^2y} - ye^x - x - y + 2 = 0$
Let $F(x,y) = xe^{x^2y} - ye^x - x - y + 2$.
$F_x(x,y) = 2x^2ye^{x^2y} + e^{x^2y} - ye^x - 1, F_y(x,y) = x^3e^{x^2y} - e^x - 1$.
Then, $\frac{dy}{dx} = -\frac{Fx}{Fy} = -\frac{2x^2ye^{x^2y} + e^{x^2y} - ye^x - 1}{x^3e^{x^2y} - e^x - 1}$.
$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} = -\frac{2e^1 + e^1 - e^1 - 1}{e^1 - e^1 - 1} = 2e - 1$ □

Q42

Solution.
Let $x = ts^2, y = \frac{s}{t}$. Then, $w = f(x,y)$.
$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial x}(x,y)(s^2) + \frac{\partial f}{\partial y}(x,y)(-\frac{s}{t^2})$
$= (xy)(s^2) + (\frac{x^2}{2})(-\frac{s}{t^2}) = (ts^2\frac{s}{t})(s^2) + (\frac{(ts^2)^2}{2})(-\frac{s}{t^2}) = s^5 - \frac{s^5}{2} = \frac{s^5}{2}$
$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(x,y)(2ts) + \frac{\partial f}{\partial y}(x,y)(\frac{1}{t})$
$= (xy)(2ts) + (\frac{x^2}{2})(\frac{1}{t}) = (ts^2\frac{s}{t})(2ts) + (\frac{(ts^2)^2}{2})(\frac{1}{t}) = 2s^4t + \frac{s^4t}{2} = \frac{5s^4t}{2}$ □

Q50

Solution.

(a) $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} = f_x \cos\theta + f_y \sin\theta$
$\frac{1}{r}\frac{\partial w}{\partial \theta} = \frac{1}{r}\left(\frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta}\right) = \frac{1}{r}\left(f_x(-r\sin\theta) + f_y r\cos\theta\right) = -f_x\sin\theta + f_y\cos\theta$

(b) $\frac{\partial w}{\partial r}\sin\theta = f_x \sin\theta\cos\theta + f_y \sin^2\theta$
$\frac{\cos\theta}{r}\frac{\partial w}{\partial \theta} = -f_x\sin\theta\cos\theta + f_y\cos^2\theta$
Then, $\frac{\partial w}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta} = f_y(\sin^2\theta + \cos^2\theta)$.
$f_y = \sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta}$
Then, $\frac{\partial w}{\partial r} = f_x\cos\theta + (\sin\theta\frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta})\sin\theta = f_x\cos\theta + (\sin^2\theta\frac{\partial w}{\partial r} + \frac{\sin\theta\cos\theta}{r}\frac{\partial w}{\partial \theta})$.
$f_x\cos\theta = (1 - \sin^2\theta)\frac{\partial w}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial w}{\partial \theta} = \cos^2\theta\frac{\partial w}{\partial r} - \frac{\sin\theta\cos\theta}{r}\frac{\partial w}{\partial \theta}$
$f_x = \cos\theta\frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$

(c) $(f_x)^2 + (f_y)^2 = (\cos\theta \frac{\partial w}{\partial r} - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta})^2 + (\sin\theta \frac{\partial w}{\partial r} + \frac{\cos\theta}{r}\frac{\partial w}{\partial \theta})^2$

$= (\cos^2\theta(\frac{\partial w}{\partial r})^2 - 2\frac{\sin\theta\cos\theta}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} + \frac{\sin^2\theta}{r^2}(\frac{\partial w}{\partial \theta})^2) + (\sin^2\theta(\frac{\partial w}{\partial r})^2 + 2\frac{\sin\theta\cos\theta}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial \theta} + \frac{\cos^2\theta}{r^2}(\frac{\partial w}{\partial \theta})^2)$

$= (\cos^2\theta + \sin^2\theta)(\frac{\partial w}{\partial r})^2 + \frac{\sin^2\theta + \cos^2\theta}{r^2}(\frac{\partial w}{\partial \theta})^2 = (\frac{\partial w}{\partial r})^2 + \frac{1}{r^2}(\frac{\partial w}{\partial \theta})^2$

□

Q60

Solution.

$F(x) = \int_{f(x)}^{a} g(t,x)dt$

Let $G(u,x) = \int_a^u g(t,x)dt$ where $u = f(x)$.

Then, $F'(x) = \frac{\partial G}{\partial x} = \frac{\partial G}{\partial u}\frac{du}{dx} + \frac{\partial G}{\partial x}\frac{dx}{dx} = g(u,x)f'(x) + \int_a^u g_x(t,x)dt$.

For $F(x) = \int_{x^2}^{1} \sqrt{t^3+x^2}dt = -\int_1^{x^2} \sqrt{t^3+x^2}dt$, let $g(t,x) = \sqrt{t^3+x^2}, u = x^2, a = 1$.

Hence, $F'(x) = -(\sqrt{u^3+x^2}(2x) + \int_1^u \frac{\partial}{\partial x}\sqrt{t^3+x^2}dt) = -(\sqrt{(x^2)^3+x^2}(2x) + \int_1^{x^2} \frac{x}{\sqrt{t^3+x^2}}dt)$

$= -2x\sqrt{x^6+x^2} + \int_{x^2}^{1} \frac{x}{\sqrt{t^3+x^2}}dt$

□

§13.6

Q4

Solution.

(a) Let $f(x,y,z) = x^2 + 2xy - y^2 + z^2$.
$f_x(x,y,z) = 2x + 2y, f_y(x,y,z) = 2x - 2y, f_z(x,y,z) = 2z$.
$f_x(1,-1,3) = 2 - 2 = 0, f_y(1,-1,3) = 2 + 2 = 4, f_z(1,-1,3) = 6$.
Then the tangent plane is given by $4(y - (-1)) + 6(z - 3) = 0 \Rightarrow 2y + 3z = 7$.

(b) Normal line : $(x,y,z) = (1, -1 + 4t, 3 + 6t)$

□

Q10

Solution.

(a) $ye^x + ze^{y^2} - z = 0$
Let $f(x,y,z) = ye^x + ze^{y^2} - z$.
$f_x(x,y,z) = ye^x, f_y(x,y,z) = e^x + 2yze^{y^2}, f_z(x,y,z) = e^{y^2} - 1$.
$f_x(0,0,1) = 0, f_y(0,0,1) = e^0 = 1, f_z(0,0,1) = e^0 - 1 = 0$.
Then the tangent plane is given by $1(y - 0) = 0 \Rightarrow y = 0$.

(b) Normal line: $(x,y,z) = (0, 0 + t, 1) = (0, t, 1)$

□